

NEF VECTOR BUNDLES ON A PROJECTIVE SPACE OR A HYPERQUADRIC WITH THE FIRST CHERN CLASS SMALL

MASAHIRO OHNO

ABSTRACT. We give a new proof of the classification due to Peternell-Szurek-Wiśniewski of nef vector bundles on a projective space with the first Chern class less than three and on a smooth hyperquadric with the first Chern class less than two over an algebraically closed field of characteristic zero.

1. INTRODUCTION

Let X be a smooth complex projective variety of dimension n , and \mathcal{E} a nef vector bundle of rank r on it. Many authors have studied a pair (X, \mathcal{E}) of X and \mathcal{E} in connection with classifications of special types of Fano (or Fano-like) manifolds, where \mathcal{E} may be assumed to be spanned (i.e., globally generated) or ample, as the case may be. In these classifications, often appears a pair (X, \mathbb{E}) of X of Picard number one and an ample vector bundle \mathbb{E} with the adjoint bundle $K_X + \det \mathbb{E}$ trivial (i.e., isomorphic to the structure sheaf). In case $K_X + \det \mathbb{E}$ is trivial, we have $r \leq n + 1$ by Mori's theory of extremal rays [9], and such pairs (X, \mathbb{E}) are classified in the cases $r = n + 1$, n , and $n - 1$ by Ye-Zhang [24], Peternell [17] [18], Fujita [5], and Peternell-Szurek-Wiśniewski [19]. Note here that the projective space bundle $\mathbb{P}(\mathbb{E})$ is a Fano manifold; in such a case, \mathbb{E} is called a Fano bundle. In this vein, in view of $-K_{\mathbb{P}(\mathbb{E})} = rH(\mathbb{E})$, \mathbb{E} might be called a “Del-Pezzo bundle” if $r = n - 1$, and a “Mukai bundle” if $r = n - 2$. As is well known, Mukai manifolds, i.e., Fano manifolds of coindex three, are described in [10], whereas the “Mukai bundle” \mathbb{E} has not been investigated for an arbitrary rank r even if the underlying manifold X is simple such as \mathbb{P}^n or \mathbb{Q}^n . Thus the present deepest result in this direction is the classification of “Del-Pezzo bundles” due to Peternell-Szurek-Wiśniewski [19]. Roughly speaking, their method of classification is to relate the pair (X, \mathbb{E}) with a pair (X, \mathcal{E}) by setting $\mathcal{E} = \mathbb{E}(-1)$, and show that \mathcal{E} is nef by their Comparison lemma [19, (3.1)]: an ample \mathbb{E} can be replaced by a nef \mathcal{E} . Then they classified such (X, \mathcal{E}) 's in [16] in the following cases: if X is isomorphic to a projective \mathbb{P}^n and the first Chern class $c_1(\mathcal{E})$ of \mathcal{E} is less than three; if X is isomorphic to a hyperquadric \mathbb{Q}^n ($n \geq 3$) and $c_1(\mathcal{E}) < 2$. (Here, for simplicity as in [16], we identify the first Chern class $c_1(\mathcal{E})$ with the corresponding non-negative integer via the isomorphism $\text{Pic } X \cong \mathbb{Z}$ by abuse of notation.) Thus the classification [16] of the pairs (X, \mathcal{E}) in the above cases has fundamental importance in their proof. Towards the classification of “Mukai bundles”, it seems therefore natural to consider the classification of (X, \mathcal{E}) in the next cases, e.g., $X = \mathbb{P}^n$ and $c_1(\mathcal{E}) = 3$, or $X = \mathbb{Q}^n$ and $c_1(\mathcal{E}) = 2$. However the classification of (X, \mathcal{E}) with \mathcal{E} nef of an arbitrary rank r in the next cases has not been pursued over twenty years.

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One of the reason why such research has not been pursued seems to come from the fact that it is uncertain how to describe nef bundles in general. In order to overcome this situation, in this paper, we first propose a framework to describe nef bundles on a projective space or a smooth hyperquadric. Following this framework, we secondly give a new proof of the above classification of (X, \mathcal{E}) due to Peternell-Szurek-Wisńiewski [16]. In their proof, Peternell-Szurek-Wisńiewski analyze the contraction morphism of an extremal ray of the Fano manifold $\mathbb{P}(\mathcal{E})$. On the other hand, our proof depends only on the cohomological study of \mathcal{E} with respect to a full strong exceptional sequence of vector bundles, and does not analyze any contraction morphism. (See Theorems 6.3, 6.5, and 9.3 and the proofs therein.)

More precisely our framework is based on the following observation. Suppose there exists a full strong exceptional sequence G_0, \dots, G_m of vector bundles on X . Recall here that a projective space or a smooth hyperquadric admits such a full strong exceptional sequences of vector bundles by [2] and [6]. Denote by G the direct sum $\bigoplus_{i=0}^m G_i$, and by A the endomorphism ring $\text{End}(G)$ of G . Let F be a coherent sheaf on X . Then Bondal's theorem [3, Theorem 6.2] implies an isomorphism

$$\text{RHom}(G, F) \otimes_A^{\mathbb{L}} G \cong F.$$

Suppose first that $\text{Ext}^q(G, F) = 0$ for all $q > 0$. Then $\text{RHom}(G, F) \cong \text{Hom}(G, F)$, and the isomorphism above implies that a projective resolution of the right A -module $\text{Hom}(G, F)$ induces the following locally free resolution of F :

$$0 \rightarrow G_0^{\oplus e_{m,0}} \rightarrow \dots \rightarrow \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}} \rightarrow \dots \rightarrow \bigoplus_{j=0}^m G_j^{\oplus e_{0,j}} \rightarrow F \rightarrow 0$$

where $e_{0,j} = \dim \text{Hom}(G_j, F)$ for all $j = 0, \dots, m$ and, for any $l \geq 1$ and any $j \leq m-l$, $e_{l,j}$ is determined inductively by the following formula: $e_{l,j} = \sum_{j < k} e_{l-1,k} \dim \text{Hom}(G_j, G_k)$. For an arbitrary coherent sheaf F , we have $\text{Ext}^q(G, F(d)) = 0$ for all $q > 0$ if d is sufficiently large by Serre's vanishing. Therefore we have a resolution of F of the above form by replacing (G_0, \dots, G_m) by a new full strong exceptional sequence $(G_0(-d), \dots, G_m(-d))$. Note here that Serre's vanishing does not give, in general, an effective estimate of the integer d . However, on a projective space or a smooth hyperquadric, the full strong exceptional sequence G_0, \dots, G_m can be chosen to be that of well-understood vector bundles so that, by applying the Kodaira or Kawamata-Viehweg vanishing theorem, we can give an *effective* estimate of the integers d such that $\text{Ext}^q(G, \mathcal{E}(d)) = 0$ for all $q > 0$ for a *nef* vector bundle \mathcal{E} (Corollaries 4.2, 4.4, and 4.6).

Let us look at our proof more closely, e.g., in the case where X is an odd dimensional smooth complex hyperquadric \mathbb{Q}^n ; in this case, we can take the sequence (G_0, \dots, G_m) to be $(\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \dots, \mathcal{O}(n-1))$ where \mathcal{S} is the (spanned) spinor bundle. For a nef vector bundle \mathcal{E} on X , let d_{\min} be the minimal integer d_{\min} such that $\text{Ext}^q(G, \mathcal{E}(d')) = 0$ for all $q > 0$ and all $d' \geq d_{\min}$. Then the Kodaira vanishing theorem shows that $d_{\min} \leq c_1(\mathcal{E})$ (Corollary 4.6), and we have the following locally free resolution

$$0 \rightarrow G_0^{\oplus e_{m,0}} \rightarrow \dots \rightarrow \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}} \rightarrow \dots \rightarrow \bigoplus_{j=0}^m G_j^{\oplus e_{0,j}} \rightarrow \mathcal{E}(d_{\min}) \rightarrow 0.$$

By tensoring $\mathcal{O}(-d_{\min})$, we get a locally free resolution of \mathcal{E} in term of a full strong exceptional sequence $(\mathcal{O}(-d_{\min}), \mathcal{S}(-d_{\min}), \mathcal{O}(1-d_{\min}), \dots, \mathcal{O}(n-d_{\min}))$. Moreover the

fact that \mathcal{E} is nef imposes several constraints on $e_{l,j}$'s and d_{\min} ; some easy constraints are that $e_{l,j} = 0$ if $l + j > d_{\min} + c_1(\mathcal{E}) + 1$ (Propositions 2.6 and 3.1 (2) (a)) and that $d_{\min} \geq 0$ if $c_1(\mathcal{E}) < r$ (Proposition 3.1 (2) (d)). Therefore if $c_1(\mathcal{E})$ is small then d_{\min} has very few possible values and most $e_{l,j}$'s vanish. Note that the above resolution contains superfluous direct summands, so that we have to remove redundant direct summands. If $c_1(\mathcal{E}) = 1$, other constraints among $e_{l,j}$'s and a more detailed analysis of the resolution enable us to do so and we get the desired resolution as described in the classification due to Peternell-Szurek-Wisniewski [16].

In the subsequent papers [11] and [12], following our framework, we classify nef vector bundles on a projective space with the first Chern class three and the second Chern class less than eight, and nef vector bundles on a smooth quadric surface with the first Chern class $(2, 1)$.

1.1. Notation and conventions. In this paper, we work over an algebraically closed field K . Basically we follow the standard notation and terminology in algebraic geometry. For example, for a vector bundle \mathcal{E} , $\mathbb{P}(\mathcal{E})$ denotes $\text{Proj } S(\mathcal{E})$, where $S(\mathcal{E})$ denotes the symmetric algebra of \mathcal{E} . The tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is also denoted by $H(\mathcal{E})$. For a coherent sheaf \mathcal{F} on a smooth projective variety X , we denote by $c_i(\mathcal{F})$ the i -th Chern class of \mathcal{F} . For a smooth projective variety X , denote by $D^b(X)$ the bounded derived category of the abelian category of coherent sheaves on X , and call $D^b(X)$ the bounded derived category of coherent sheaves on X for short. We say that a vector bundle is spanned or globally generated if it is generated by global sections. For “spinor bundles”, we follow Kapranov’s convention [6]; our spinor bundles are spanned and they are the duals of those of Ottaviani’s [15]. See [13, §5 Definition 1] for a precise definition of our spinor bundles. Finally we refer to [8] for the definition and basic properties of nef vector bundles.

2. PRELIMINARIES

Let X be an n -dimensional smooth projective variety over K , and suppose that there exists a full strong exceptional sequence G_0, \dots, G_m of vector bundles on X .

Recall here the definition of a full strong exceptional sequence. An object G_i of the bounded derived category $D^b(X)$ of coherent sheaves on X is said to be exceptional if $\text{RHom}(G_i, G_i) \cong K$ and a sequence G_0, \dots, G_m of exceptional objects is said to be exceptional if $\text{RHom}(G_i, G_j) = 0$ for all $0 \leq j < i \leq m$. An exceptional sequence G_0, \dots, G_m is said to be strong if $\text{Ext}^k(G_i, G_j) = 0$ for all $k > 0$ and $0 \leq i < j \leq m$. Finally a strong exceptional sequence G_0, \dots, G_m is said to be full if $D^b(X)$ is the smallest triangulated full subcategory containing G_0, \dots, G_m and closed under isomorphism.

If X is an n -dimensional projective space \mathbb{P}^n , then $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$ is a strong exceptional sequence of line bundles, and it is full by Beilinson’s theorem [2, Theorem]. If X is an odd-dimensional smooth hyperquadric \mathbb{Q}^n and the characteristic $\text{char } K$ of the base field K is zero, then it follows from Bott’s vanishing theorem that $(\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \dots, \mathcal{O}(n-1))$ is a strong exceptional sequence of vector bundles, where \mathcal{S} is the spinor bundle, and it is full by Kapranov’s theorem [6, Theorem 4.10]. If X is an even-dimensional smooth hyperquadric \mathbb{Q}^n and $\text{char } K = 0$, then Bott’s vanishing theorem shows that $(\mathcal{O}, \mathcal{S}^+, \mathcal{S}^-, \mathcal{O}(1), \dots, \mathcal{O}(n-1))$ is a strong exceptional sequence of vector bundles, where \mathcal{S}^+ and \mathcal{S}^- are spinor bundles, and it is full by Kapranov’s theorem [6, Theorem 4.10]. Recall here that we follow Kapranov’s convention for “spinor bundles”; for example, on a

smooth quadric surface \mathbb{Q}^2 , spanned line bundles $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ are spinor bundles, and thus $(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1))$ is a full strong exceptional sequence on \mathbb{Q}^2 .

For some other fundamental facts about derived categories, we refer to an excellent book [7] of Kashiwara-Schapira as a literature written in English.

Denote by G the direct sum $\bigoplus_{i=0}^m G_i$ of G_0, \dots, G_m , and by A the endomorphism ring $\text{End}(G)$ of G . Then A is a finite-dimensional K -algebra. We refer to [1, Chap. I, II, III] for some basic facts about modules over a finite-dimensional K -algebra. Note that we follow the convention that the composite of two arrows $\alpha : a \rightarrow b$ and $\beta : b \rightarrow c$ is denoted by $\beta\alpha$. In the same vein, we regard G as a left A -module.

For a coherent sheaf F on X , Bondal's theorem [3, Theorem 6.2] implies that

$$\text{RHom}(G, F) \otimes_A^{\mathbb{L}} F \cong F,$$

so that if $\text{Ext}^q(G, F) = 0$ for all $q > 0$ then $\text{RHom}(G, F) \cong \text{Hom}(G, F)$, and a projective resolution of the right A -module $\text{Hom}(G, F)$ will play a key role in this paper; let us recall here briefly a projective resolution of a right A -module.

Let $p_i : G \rightarrow G_i$ be the projection, and $\iota_i : G_i \hookrightarrow G$ the inclusion. Set $e_i = \iota_i \circ p_i$ in A . Denote $e_i A$ by P_i . Then $P_i \cong \text{Hom}(G, G_i)$ as right A -modules, and $A = \bigoplus_{i=0}^m P_i$; P_i is projective and $P_i \otimes_A G \cong G_i$.

For a finitely generated right A -module V , a right A -submodule $V^{\leq i}$ of V is defined by the formula $V^{\leq i} = \bigoplus_{j \leq i} V e_j$. We have a natural isomorphism $V^{\leq i} \cong V \otimes_A A^{\leq i}$, and associated to every module V is an ascending filtration

$$0 = V^{\leq -1} \subset V^{\leq 0} \subset V^{\leq 1} \subset \dots \subset V^{\leq m} = V$$

by right A -submodules. Set $\text{Gr}^i V = V^{\leq i} / V^{\leq i-1}$; $\text{Gr}^i V$ is a right A -module. Denote by V^i the K -vector subspace $V e_i$ of V . Note that V^i is not a A -submodule of V , but we have an isomorphism $\text{Gr}^i V \cong V^i$ of K -vector spaces. For example, we have

$$P_k^{\leq i} \cong \bigoplus_{j \leq i} \text{Hom}(G_j, G_k) \text{ and } P_k^j \cong \text{Hom}(G_j, G_k).$$

Note in particular that $P_k^k \cong K$ and that $P_k^j = 0$ if $j > k$. For a homomorphism $\varphi : V \rightarrow W$ of right A -modules, we denote by φ^i the induced homomorphism $V^i \rightarrow W^i$ of K -vector spaces.

We have a natural right A -linear map

$$\varphi_{i,V} : V^i \otimes_K P_i \rightarrow V$$

sending $v \otimes a$ to va . Since every element v of V^i can be written as $v = v' e_i$ for some $v' \in V$, we see $\varphi_{i,V}(v \otimes e_i) = v$. Hence the induced K -linear $\varphi_{i,V}^i$ is an isomorphism:

$$(V^i \otimes_K P_i)^i \cong V^i \otimes_K P_i^i \cong V^i.$$

All $\varphi_{i,V}$ together give a canonical surjection

$$\varphi_V : \bigoplus_j V^j \otimes_K P_j \rightarrow V.$$

Set $W = \text{Ker } \varphi_V$, and consider the canonical surjection

$$\varphi_W : \bigoplus_i W^i \otimes_K P_i \rightarrow W$$

for W . Here, for a non-zero V , define $d(V)$ by the following formula:

$$d(V) = \max\{j \in \mathbb{Z}_{\geq 0} \mid V^j \neq 0\}.$$

Since φ_V^i is a surjective K -linear $\oplus_{i \leq j} V^j \otimes_K P_j^i \cong (\oplus_{j \leq m} V^j \otimes_K P_j)^i \rightarrow V^i$, we see that

$$\dim W^i = \sum_{i < j \leq d(V)} \dim V^j \dim P_j^i = \sum_{i < j \leq d(V)} \dim V^j \dim \operatorname{Hom}(G_i, G_j).$$

In particular, $W^i = 0$ for all $i \geq d(V)$. These consideration leads to the following.

Lemma 2.1. *Every finitely generated right A -module V has a bounded projective resolution of the following form*

$$0 \rightarrow P_0^{\oplus e_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} P_j^{\oplus e_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m P_j^{\oplus e_{0,j}} \rightarrow V \rightarrow 0$$

where $e_{0,j} = \dim V^j$ for all $j = 0, \dots, m$ and, for any $l \geq 1$ and any $j \leq m-l$, $e_{l,j}$ is determined inductively by the following formula: $e_{l,j} = \sum_{j < k} e_{l-1,k} \dim \operatorname{Hom}(G_j, G_k)$.

Remark 2.2. *The resolution above is not minimal in general. Throughout this paper, we shall denote by*

$$0 \rightarrow P_0^{\oplus e'_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} P_j^{\oplus e'_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m P_j^{\oplus e'_{0,j}} \rightarrow V \rightarrow 0$$

a minimal resolution of V with $0 \leq e'_{l,j} \leq e_{l,j}$ for all $0 \leq j \leq m-l \leq m$.

Proposition 2.3. *Under the assumption and notation as above, let F be a coherent sheaf on X . Suppose that $\operatorname{Ext}^q(G, F) = 0$ for all $q > 0$. Then F has a locally free resolution of the following form:*

$$0 \rightarrow G_0^{\oplus e_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m G_j^{\oplus e_{0,j}} \rightarrow F \rightarrow 0$$

where $e_{0,j} = \dim \operatorname{Hom}(G_j, F)$ for all $j = 0, \dots, m$ and, for any $l \geq 1$ and any $j \leq m-l$, $e_{l,j}$ is determined inductively by the following formula:

$$e_{l,j} = \sum_{j < k} e_{l-1,k} \dim \operatorname{Hom}(G_j, G_k).$$

Proof. Since $\operatorname{Ext}^q(G, F) = 0$ for all $q > 0$, Bondal's theorem [3, Theorem 6.2] implies that $\operatorname{Hom}(G, F) \otimes_A^{\mathbb{L}} G \cong F$. Since $\operatorname{Hom}(G, F)^j \cong \operatorname{Hom}(G_j, F)$, Lemma 2.1 shows the following projective resolution of $\operatorname{Hom}(G, F)$:

$$0 \rightarrow P_0^{\oplus e_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} P_j^{\oplus e_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m P_j^{\oplus e_{0,j}} \rightarrow \operatorname{Hom}(G, F) \rightarrow 0.$$

Since $P_j \otimes_A G \cong G_j$, the projective resolution above induces the desired locally free resolution of the coherent sheaf F . \square

Set $\mathcal{P}_l = \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}}$ for $0 \leq l \leq m$, and let \mathcal{P}_\bullet denote the resulting complex.

Remark 2.4. Set $\mathrm{HN}_i(\mathcal{P}_\bullet) = \bigoplus_{j=i}^{m-\bullet} G_j^{\oplus e_{\bullet,j}}$. Then \mathcal{P}_\bullet has the following filtration of Harder-Narasimhan type

$$0 \rightarrow \mathrm{HN}_m(\mathcal{P}_\bullet) \rightarrow \cdots \rightarrow \mathrm{HN}_i(\mathcal{P}_\bullet) \rightarrow \cdots \rightarrow \mathrm{HN}_0(\mathcal{P}_\bullet) = \mathcal{P}_\bullet$$

with $\mathrm{HN}_i(\mathcal{P}_\bullet)/\mathrm{HN}_{i+1}(\mathcal{P}_\bullet) \cong \bigoplus_{i \leq m-\bullet} G_i^{\oplus e_{\bullet,i}} \in \langle G_i \rangle$, where $\langle G_i \rangle$ denotes the smallest triangulated subcategory of $D^b(X)$ containing G_i and closed under isomorphism. Note that $\mathrm{Hom}_{D^b(X)}(\langle G_i \rangle, \langle G_j \rangle) = 0$ if $i > j$. If we regard \mathcal{P}_\bullet and F as objects of $D^b(X)$ and the filtration above as that in $D^b(X)$, then $\mathcal{P}_\bullet \cong F$ and semiorthogonality of $\langle G_0 \rangle, \dots, \langle G_m \rangle$ implies that the filtration above is unique and functorial with respect to F .

Let \mathcal{E} be a coherent sheaf on X , and let $\mathcal{O}_X(1)$ be an ample line bundle on X . By Serre's vanishing theorem, we see that if $d' \gg 0$ then $\mathrm{Ext}^q(G, \mathcal{E}(d')) = 0$ for all $q > 0$. Let d_{\min} be the minimal integer d_{\min} such that $\mathrm{Ext}^q(G, \mathcal{E}(d')) = 0$ for all $q > 0$ and all $d' \geq d_{\min}$.

Corollary 2.5. Under the notation above, $\mathcal{E}(d_{\min})$ fits in the following exact sequence:

$$0 \rightarrow G_0^{\oplus e_{m,0}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^{m-l} G_j^{\oplus e_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=0}^m G_j^{\oplus e_{0,j}} \rightarrow \mathcal{E}(d_{\min}) \rightarrow 0,$$

where $e_{0,j} = \dim \mathrm{Hom}(G_j, \mathcal{E}(d_{\min}))$ for all $j = 0, \dots, m$ and, for any $l \geq 1$ and any $j \leq m-l$, $e_{l,j}$ is determined inductively by $e_{l,j} = \sum_{j < k} e_{l-1,k} \dim \mathrm{Hom}(G_j, G_k)$. Moreover we can replace $e_{l,j}$ by some integer $e'_{l,j}$ such that $0 \leq e'_{l,j} \leq e_{l,j}$ for all $0 \leq j \leq m-l \leq m$ corresponding to the minimal resolution (see Remark 2.2 for the precise definition of $e'_{l,j}$).

In the rest of this paper, we call the exact sequence in Corollary 2.5 the *standard resolution of $\mathcal{E}(d_{\min})$ with respect to the (prescribed) full strong exceptional sequence (G_0, \dots, G_m) of vector bundles*, and let $e'_{l,j}$ and $e_{l,j}$ be as in Corollary 2.5. The following is an easy but fundamental relation among $e_{l,j}$'s.

Proposition 2.6. If $e_{l,k} = 0$ for all $k > j$, then $e_{l+1,k} = 0$ for all $k > j-1$.

Proof. This follows immediately from the definition of $e_{l,j}$. \square

To reduce $e_{0,0}$ to $e'_{0,0}$, the following proposition is also fundamental.

Proposition 2.7. If $\mathcal{E}(d_{\min})$ does not admit G_0 as a quotient. Then $e_{1,0} \geq e_{0,0}$ and, in the standard resolution, we can replace \mathcal{P}_0 by \mathcal{P}_0^0 and \mathcal{P}_1 by \mathcal{P}_1^0 , where $\mathcal{P}_0^0 = \bigoplus_{j=1}^m G_j^{\oplus e_{0,j}}$ and $\mathcal{P}_1^0 = G_0^{\oplus e_{1,0} - e_{0,0}} \oplus (\bigoplus_{j=1}^{m-1} G_j^{\oplus e_{1,j}})$. In particular we see that $e'_{0,0} = 0$.

Proof. Denote by d_1 the differential $\mathcal{P}_1 \rightarrow \mathcal{P}_0$, and let $p_{l,0} : \mathcal{P}_l \rightarrow G_0^{\oplus e_{l,0}}$ be the projection. Then the composite $p_{0,0} \circ d_1$ factors through $p_{1,0}$: $p_{0,0} \circ d_1 = d_{1,0} \circ p_{1,0}$ where $d_{1,0} : G_0^{\oplus e_{1,0}} \rightarrow G_0^{\oplus e_{0,0}}$ is the induced morphism. Suppose that $d_{1,0}$ is not surjective. Then we have a surjection $q : G_0^{\oplus e_{0,0}} \rightarrow G_0$ such that $q \circ d_{1,0} = 0$. Since $(q \circ p_{0,0}) \circ d_1 = q \circ d_{1,0} \circ p_{1,0} = 0$, there exists a surjection $\mathcal{E}(d_{\min}) \rightarrow G_0$, which contradicts the assumption. Therefore $d_{1,0}$ is surjective and $e_{1,0} \geq e_{0,0}$. Moreover $p_{0,0} \circ d_1$ is surjective. Since $\mathrm{Ker}(p_{0,0}) = \mathcal{P}_0^0$ and $\mathrm{Ker}(p_{0,0} \circ d_1) = \mathcal{P}_1^0$, the desired replacement can be done in the standard resolution of $\mathcal{E}(d_{\min})$. \square

Remark 2.8. If $d_{\min} > 0$, $G_0 = \mathcal{O}$, and \mathcal{E} is nef, then $\mathcal{E}(d_{\min})$ does not admit the sheaf \mathcal{O} as a quotient.

3. SOME EASY CONSTRAINTS

Let $X, G_0, \dots, G_m, \mathcal{E}, \mathcal{O}_X(1), d_{\min}, e_{l,j}, e'_{l,j}$, and \mathcal{P}_l be as in § 2 for $0 \leq j \leq l \leq m$. In the rest of this paper, we always assume that \mathcal{E} is a nef vector bundle of rank r and that $\mathcal{O}_X(1)$ is a “suitable” ample line bundle on X , e.g., an ample line bundle of “minimal degree”. Then it is natural to consider the following

Problem 3.1. *What constraints does the condition that \mathcal{E} is nef impose on (or among) $d_{\min}, e_{l,j}$ ’s, and $e'_{l,j}$ ’s? Find good constraints on (or among) them.*

The rest of this paper addresses this problem in the following cases:

- (1) X is a projective space \mathbb{P}^n , $\mathcal{O}_X(1)$ is the ample generator of $\text{Pic } X$, and (G_0, \dots, G_m) is equal to $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$;
- (2) X is an odd dimensional smooth hyperquadric \mathbb{Q}^n with $n \geq 3$, the field K is of characteristic zero, $\mathcal{O}_X(1)$ is the ample generator of $\text{Pic } X$, and (G_0, \dots, G_m) is equal to $(\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \dots, \mathcal{O}(n-1))$, where \mathcal{S} is the spinor bundle on \mathbb{Q}^n ;
- (3) X is an even dimensional smooth hyperquadric \mathbb{Q}^n , the field K is of characteristic zero, and (G_0, \dots, G_m) is equal to $(\mathcal{O}, \mathcal{S}^+, \mathcal{S}^-, \mathcal{O}(1), \dots, \mathcal{O}(n-1))$, where \mathcal{S}^+ and \mathcal{S}^- are the spinor bundles on \mathbb{Q}^n . If $n \geq 4$, then $\mathcal{O}_X(1)$ is the ample generator of $\text{Pic } X$, and if $n = 2$, then $\mathcal{O}_X(1)$ is the ample line bundle $\mathcal{O}(1, 1)$ of minimal degree.

Thus we always assume, in the rest of the paper, that if X is as in (1), (2), or (3), then (G_0, \dots, G_m) is as in (1), (2), or (3) respectively.

If $\text{Pic } X \cong \mathbb{Z}$, we denote by d the integer such that $\mathcal{O}_X(d) \cong \det \mathcal{E}$, and if $X \cong \mathbb{Q}^2$, we denote by (a, b) the pair of integers such that $\mathcal{O}_{\mathbb{Q}^2}(a, b) \cong \det \mathcal{E}$.

In this section, we give some easy constraints among d_{\min} and $e_{l,j}$ ’s in the cases (1), (2), and (3) above.

Proposition 3.1. *We have the following constraints*

- (1) Suppose that $X = \mathbb{P}^n$.
 - (a) If $j > d + d_{\min}$, then $e_{0,j} = 0$.
 - (b) If $d < r$, then $d_{\min} \geq 0$.
- (2) Suppose that $X = \mathbb{Q}^n$.
 - (a) Suppose that n is odd. If $j > d + d_{\min} + 1$, then $e_{0,j} = 0$.
 - (b) Suppose that n is even. If $j > d + d_{\min} + 2$, then $e_{0,j} = 0$.
 - (c) Suppose that n is even. If $e_{l,k} = 0$ for all $k > 2$, then $e_{l+1,k} = 0$ for all $k > 0$. In particular $e_{n,1} = 0$ and $e_{n+1,0} = 0$.
 - (d) If $d < r$, then $d_{\min} \geq 0$.
 - (e) Suppose that $n = 2$. If $\min\{a, b\} < r$, then $d_{\min} \geq 0$.

Proof. The proofs of (1) (a), (2) (a), and (2) (b) are essentially the same; the only difference comes from the fact that there exists \mathcal{S} or a pair of \mathcal{S}^+ and \mathcal{S}^- between \mathcal{O} and $\mathcal{O}(1)$, so that the numbering of G_j differs. For simplicity, we only write explicitly the proof of (1) (a), but the reader will easily modify the proof for (2) (a), and (2) (b).

(1) (a) Suppose that $e_{0,j} \neq 0$ for some $j > d + d_{\min}$. Then we have a non-zero map $\mathcal{O}(j) \rightarrow \mathcal{E}(d_{\min})$, which gives a non-zero map $\mathcal{O}_L(j) \rightarrow \mathcal{E}|_L(d_{\min})$ for a general line L in \mathbb{P}^n . This implies that the maximal degree of a direct summand of $\mathcal{E}|_L(d_{\min})$ is at least j . On the other hand, since \mathcal{E} is nef, the maximal degree of a direct summand of $\mathcal{E}|_L(d_{\min})$ is at most $d + d_{\min}$. This is a contradiction, since $j > d + d_{\min}$.

(2) (c) This is because $e_{l+1,1} = \sum_{1 \leq k} e_{l,k} \dim \operatorname{Hom}(G_1, G_k) = e_{l,2} \dim \operatorname{Hom}(\mathcal{S}^+, \mathcal{S}^-)$ and $\operatorname{Hom}(\mathcal{S}^+, \mathcal{S}^-) = 0$.

The proofs of (1) (b), (2) (d), and (2) (e) are essentially the same; For simplicity, we only write explicitly the proof of (2) (d).

(2) (d) We have a surjection $\mathcal{P}_0 \rightarrow \mathcal{E}(d_{\min})$. Since \mathcal{P}_0 is globally generated, the restriction $\mathcal{E}|_L(d_{\min})$ to a line L in \mathbb{Q}^n is also globally generated. Hence the minimal degree of a direct summand of $\mathcal{E}|_L(d_{\min})$ is non-negative. Note here that the minimal degree of a direct summand of $\mathcal{E}|_L(d_{\min})$ is d_{\min} since $r > d$. Therefore $d_{\min} \geq 0$. \square

4. AN UPPER BOUND FOR d_{\min}

Let $X, G_0, \dots, G_m, G, \mathcal{E}, \mathcal{O}_X(1)$, and d_{\min} be as in § 2. Assume that \mathcal{E} be a nef vector bundle of rank r as in § 3. In this section, we assume that the base field K is of characteristic zero, and we give an upper bound for d_{\min} in the cases (1), (2), and (3) as described in § 3. Let $\mathcal{O}_X(1)$, d , and (a, b) be as in § 3.

Let $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ be the projection, and let $H(\mathcal{E})$ be the tautological line bundle on $\mathbb{P}(\mathcal{E})$.

In Lemma 4.1 below, in order to unify and shorten descriptions in \mathbb{P}^n and \mathbb{Q}^n ($n \geq 3$), we denote by $c_1(X)$ the integer corresponding to $c_1(X)$ via the isomorphism $\operatorname{Pic} X \cong \mathbb{Z}$ which sends the ample generator to 1. However, needless to say, we must not use this abuse of notation in intersection formulas.

Lemma 4.1. *Let X be a smooth Fano variety of dimension n with $\operatorname{Pic} X \cong \mathbb{Z}$, and let $\mathcal{O}_X(1)$ be the ample generator of $\operatorname{Pic} X$. Then we have the following vanishing.*

- (1) $\operatorname{Ext}^q(\mathcal{O}(j), \mathcal{E}(d)) = 0$ for all $q > 0$ and $j < c_1(X)$.
- (2) $\operatorname{Ext}^q(\mathcal{O}(c_1(X)), \mathcal{E}(d)) = 0$ for all $q > 0$ if $H(\mathcal{E})^{n+r-1} > 0$. If $n = 2$ then the condition $H(\mathcal{E})^{n+r-1} > 0$ is equivalent to the one that $c_1(\mathcal{E})^2 - c_2(\mathcal{E}) > 0$.

Proof. (1) We have isomorphisms

$$\operatorname{Ext}^q(\mathcal{O}(j), \mathcal{E}(d)) \cong H^q(X, \mathcal{E}(d-j)) \cong H^q(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + \pi^* \mathcal{O}_X(d-j)).$$

We claim that the last cohomology group vanishes by the Kodaira vanishing theorem; indeed, since $-K_X \cong \mathcal{O}_X(c_1(X))$, we have

$$H(\mathcal{E}) + \pi^* \mathcal{O}_X(d-j) - K_{\mathbb{P}(\mathcal{E})} \cong (r+1)H(\mathcal{E}) + \pi^* \mathcal{O}_X(c_1(X) - j),$$

and $(r+1)H(\mathcal{E}) + \pi^* \mathcal{O}_X(c_1(X) - j)$ is ample by the Nakai-Moishezon criterion, since $j < c_1(X)$. Therefore the claim follows.

(2) If $j = c_1(X)$, then $H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{P}^n}(d-j) - K_{\mathbb{P}(\mathcal{E})}$ is isomorphic to $(r+1)H(\mathcal{E})$, and this is nef and big by assumption. The result then follows from the Kawamata-Viehweg vanishing theorem. The assertion for $n = 2$ follows from $H(\mathcal{E})^{r+1} = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$ if $n = 2$. \square

Corollary 4.2. *Let \mathcal{E} be a nef vector bundle of rank r on \mathbb{P}^n . Then $d_{\min} \leq d$. Moreover if $H(\mathcal{E})^{n+r-1} > 0$ then $d_{\min} < d$. In particular if $n = 2$ and $c_1(\mathcal{E})^2 - c_2(\mathcal{E}) > 0$, then $d_{\min} < d$.*

Lemma 4.3. *Suppose that X is a smooth hyperquadric \mathbb{Q}^n of dimension $n \geq 3$. Then $\operatorname{Ext}^q(\mathcal{S}, \mathcal{E}(d+j)) = 0$ for all $q > 0$ and $j \geq -\lfloor \frac{n}{2} \rfloor + 1$, where \mathcal{S} is a spinor bundle on \mathbb{Q}^n .*

Proof. We have an isomorphism $\text{Ext}^q(\mathcal{S}, \mathcal{E}(d+j)) \cong H^q(\mathbb{Q}^n, \mathcal{S}^\vee \otimes \mathcal{E}(d+j))$. By Theorem 8.1 (1), (2), and (3), to show $H^q(\mathbb{Q}^n, \mathcal{S}^\vee \otimes \mathcal{E}(d+j)) = 0$ for all $q > 0$ and $j \geq -\lfloor \frac{n}{2} \rfloor + 1$ and a spinor bundle \mathcal{S} on \mathbb{Q}^n , it is enough to show that $H^q(\mathbb{Q}^n, \mathcal{S} \otimes \mathcal{E}(d+j)) = 0$ for all $q > 0$ and $j \geq -\lfloor \frac{n}{2} \rfloor$ and a spinor bundle \mathcal{S} on \mathbb{Q}^n . We have an isomorphism $H^q(\mathbb{Q}^n, \mathcal{S} \otimes \mathcal{E}(d+j)) \cong H^q(\mathbb{P}(\mathcal{S}), H(\mathcal{S}) \otimes p^*(\mathcal{E}(d+j)))$, where $p : \mathbb{P}(\mathcal{S}) \rightarrow \mathbb{Q}^n$ is the projection and $H(\mathcal{S})$ is the tautological line bundle on $\mathbb{P}(\mathcal{S})$. Let $\tilde{\pi} : \mathbb{P}(p^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{S})$ be the projection. Since $H(\mathcal{S}) \otimes p^*(\mathcal{E}(d+j)) \cong p^*\mathcal{E} \otimes H(\mathcal{S}) \otimes p^*\mathcal{O}_{\mathbb{Q}^n}(d+j)$, we have an isomorphism

$$H^q(\mathbb{P}(\mathcal{S}), H(\mathcal{S}) \otimes p^*(\mathcal{E}(d+j))) \cong H^q(\mathbb{P}(p^*\mathcal{E}), H(p^*\mathcal{E}) + \tilde{\pi}^*(H(\mathcal{S}) + p^*\mathcal{O}_{\mathbb{Q}^n}(d+j))).$$

We claim here that the last cohomology group vanishes by the Kodaira vanishing theorem; first observe that $H(p^*\mathcal{E}) + \tilde{\pi}^*(H(\mathcal{S}) + p^*\mathcal{O}_{\mathbb{Q}^n}(d+j)) - K_{\mathbb{P}(p^*\mathcal{E})}$ is isomorphic to

$$(r+1)H(p^*\mathcal{E}) + \tilde{\pi}^*(H(\mathcal{S}) + p^*\mathcal{O}_{\mathbb{Q}^n}(j) - K_{\mathbb{P}(\mathcal{S})}).$$

To show the last line bundle is ample, it is enough by the Nakai-Moishezon criterion to show that $H(\mathcal{S}) + p^*\mathcal{O}_{\mathbb{Q}^n}(j) - K_{\mathbb{P}(\mathcal{S})}$ is ample. To see this, recall that $\mathbb{P}(\mathcal{S})$ is a flag manifold parameterizing flags of one-dimensional and maximal dimensional linear subspaces of \mathbb{Q}^n ; set $s = \lfloor \frac{n}{2} \rfloor$, and let $q : \mathbb{P}(\mathcal{S}) \rightarrow S$ be the projection, which is a \mathbb{P}^s -bundle, to the spinor variety S . Recall also that $H(\mathcal{S}) \cong q^*\mathcal{O}_S(1)$ for the ample generator $\mathcal{O}_S(1)$ of $\text{Pic } S$ (see, e.g., [13, §5]). We see that $\mathbb{P}(\mathcal{S})$ is a Fano manifold of Picard number two, that $H(\mathcal{S}) + p^*\mathcal{O}_{\mathbb{Q}^n}(j) - K_{\mathbb{P}(\mathcal{S})}$ is p -ample, and that it is q -ample if $j + s \geq 0$. Therefore $H(\mathcal{S}) + p^*\mathcal{O}_{\mathbb{Q}^n}(j) - K_{\mathbb{P}(\mathcal{S})}$ is ample if $j \geq -s$, and the claim follows. \square

Corollary 4.4. *Let \mathcal{E} be a nef vector bundle of rank r on a smooth hyperquadric \mathbb{Q}^n of dimension $n \geq 3$. Then $d_{\min} \leq d$. Moreover if $n \geq 4$ and $H(\mathcal{E})^{n+r-1} > 0$ then $d_{\min} < d$.*

Proof. This follows from Lemmas 4.1 and 4.3. \square

Finally we deal with the case $X = \mathbb{Q}^2$. We have $\text{Ext}^q(G, \mathcal{E}(d', d')) = 0$ for $d' \gg 0$ and $q > 0$ by Serre's vanishing theorem. Note that if (d_1, d_2) is a pair of integers such that $\text{Ext}^q(G, \mathcal{E}(d_1, d_2)) = 0$ for all $q > 0$, then $\mathcal{E}(d_1, d_2)$ has the standard resolution with respect to \mathcal{O} , $\mathcal{O}(1, 0)$, $\mathcal{O}(0, 1)$, and $\mathcal{O}(1, 1)$, which implies that $\text{Ext}^q(G, \mathcal{E}(d'_1, d'_2)) = 0$ for all $q > 0$, all $d'_1 \geq d_1$, and all $d'_2 \geq d_2$. Then we define a pair $(d_{1,\min}, d_{2,\min})$ of integers by the following property:

$$\begin{aligned} \text{Ext}^q(G, \mathcal{E}(d'_1, d'_2)) &= 0 \text{ for all } q > 0, \text{ all } d'_1 \geq d_{1,\min}, \text{ and all } d'_2 \geq d_{2,\min}, \\ \text{Ext}^q(G, \mathcal{E}(d_{1,\min} - 1, d_{2,\min})) &\neq 0 \text{ for some } q > 0, \\ \text{Ext}^q(G, \mathcal{E}(d_{1,\min}, d_{2,\min} - 1)) &\neq 0 \text{ for some } q > 0. \end{aligned}$$

Lemma 4.5. *Suppose that $X = \mathbb{Q}^2$. Let \mathcal{S} be a spinor bundle $\mathcal{O}(1, 0)$ or $\mathcal{O}(0, 1)$. Then we have the following vanishing.*

- (1) $\text{Ext}^q(\mathcal{O}(j, j), \mathcal{E}(a, b)) = 0$ for all $q > 0$ and $j < 2$.
- (2) $\text{Ext}^q(\mathcal{O}(2, 2), \mathcal{E}(a, b)) = 0$ for all $q > 0$, if $2ab > c_2(\mathcal{E})$.
- (3) $\text{Ext}^q(\mathcal{S}, \mathcal{E}(a+j, b+j)) = 0$ for all $q > 0$ and $j \geq 0$.
- (4) $\text{Ext}^q(\mathcal{S}(1, 1), \mathcal{E}(a, b)) = 0$ for all $q > 0$, if $2ab > c_2(\mathcal{E})$.

Proof. (1) We have isomorphisms

$$\text{Ext}^q(\mathcal{O}(j, j), \mathcal{E}(a, b)) \cong H^q(\mathcal{E}(a-j, b-j)) \cong H^q(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + \pi^*\mathcal{O}_{\mathbb{Q}^2}(a-j, b-j)).$$

We claim that the last cohomology group vanishes by the Kodaira vanishing theorem; indeed we have

$$H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(a - j, b - j) - K_{\mathbb{P}(\mathcal{E})} \cong (r + 1)H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(2 - j, 2 - j)$$

and $(r + 1)H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(2 - j, 2 - j)$ is ample by the Nakai-Moishezon criterion since $j < 2$. Therefore the claim follows.

(2) Note that $H(\mathcal{E})^{r+1} = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 2ab - c_2(\mathcal{E})$. Therefore if $2ab > c_2(\mathcal{E})$ then $H(\mathcal{E})$ is nef and big. Hence $H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(a - j, b - j) - K_{\mathbb{P}(\mathcal{E})}$ is nef and big if $j = 2$. The result then follows from the Kawamata-Viehweg vanishing theorem.

(3) Suppose that $\mathcal{S} \cong \mathcal{O}(1, 0)$. We have isomorphisms

$$\begin{aligned} \text{Ext}^q(\mathcal{S}, \mathcal{E}(a + j, b + j)) &\cong H^q(\mathbb{Q}^2, \mathcal{E}(a + j - 1, b + j)) \\ &\cong H^q(\mathbb{P}(\mathcal{E}), H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(a + j - 1, b + j)). \end{aligned}$$

We show that the last cohomology group vanishes by the Kodaira vanishing theorem; we see that $H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(a + j - 1, b + j) - K_{\mathbb{P}(\mathcal{E})}$ is isomorphic to $(r + 1)H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(j + 1, j + 2)$, and this line bundle is ample if $j \geq 0$ by the Nakai-Moishezon criterion.

(4) The proof is almost the same as (3); if $j = -1$, then $(r + 1)H(\mathcal{E}) + \pi^* \mathcal{O}_{\mathbb{Q}^2}(j + 1, j + 2)$ is nef and big if so is $H(\mathcal{E})$. Now the result follows from the Kawamata-Viehweg vanishing theorem. \square

Corollary 4.6. *Let \mathcal{E} be a nef vector bundle of rank r on \mathbb{Q}^2 . Then we can take $(d_{1,\min}, d_{2,\min})$ such that $d_{1,\min} \leq a$ and $d_{2,\min} \leq b$. Moreover we can take $(d_{1,\min}, d_{2,\min})$ such that $d_{1,\min} \leq a - 1$ and $d_{2,\min} \leq b - 1$, if $2ab > c_2(\mathcal{E})$.*

5. MAXIMAL DEGREE SUBBUNDLES OF A NEF VECTOR BUNDLE

Let X be as in § 2, and let \mathcal{E} be as in § 2. Assume that \mathcal{E} is a nef vector bundle of rank r as in § 3.

Lemma 5.1. *Suppose that there exists a non-zero morphism $\varphi : \det \mathcal{E} \rightarrow \mathcal{E}$. Then φ makes $\det \mathcal{E}$ a subbundle of \mathcal{E} .*

Proof. Let s be a non-zero element of $H^0(\mathcal{E} \otimes (\det \mathcal{E})^\vee)$ corresponding to φ , and suppose that the zero locus $(s)_0$ of s is not empty. Take a curve C such that $C \cap (s)_0 \neq \emptyset$ and that C is not contained in $(s)_0$, and let $\pi : \tilde{C} \rightarrow C$ be the normalization. Then $\mathcal{O}_{\tilde{C}}(\pi^*((s)_0 \cap C))$ is a subbundle of $\pi^*(\mathcal{E} \otimes (\det \mathcal{E})^\vee)$. This implies that $\pi^* \mathcal{E}$ has a quotient bundle of negative degree, which contradicts that \mathcal{E} is nef. Therefore $(s)_0$ is empty and φ makes $\det \mathcal{E}$ a subbundle of \mathcal{E} . \square

Proposition 5.2. *Suppose that $H^1(\det \mathcal{E}) = 0$ and that every nef vector bundle with trivial determinant is isomorphic to a direct sum of copies of \mathcal{O} . Then $\text{Hom}(\det \mathcal{E}, \mathcal{E}) \neq 0$ implies that $\mathcal{E} \cong \mathcal{O}^{\oplus r-1} \oplus \det \mathcal{E}$.*

Proof. If $\text{Hom}(\det \mathcal{E}, \mathcal{E}) \neq 0$, then, by Lemma 5.1, there exists an exact sequence

$$0 \rightarrow \det \mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

with \mathcal{F} a vector bundle. Since $\det \mathcal{F} \cong \mathcal{O}$, the assumption implies that $\mathcal{F} \cong \mathcal{O}^{\oplus r-1}$. Since $H^1(\det \mathcal{E}) = 0$, this implies that $\mathcal{E} \cong \mathcal{O}^{\oplus r-1} \oplus \det \mathcal{E}$. \square

Remark 5.3. *The assumption of Proposition 5.2 is satisfied if X is either a projective space \mathbb{P}^n or a hyperquadric \mathbb{Q}^n . See, e.g., [14, Chap. 1 Theorem 3.2.1] and [23, Lemma 3.6.1]*

Lemma 5.4. *Let \mathcal{F} be a locally free coherent sheaf, \mathcal{G} a torsion-free coherent sheaf, and let*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

be an exact sequence of coherent sheaves on X . If the support Z of torsion subsheaf \mathcal{T} of \mathcal{H} has codimension ≥ 2 in X , then $\mathcal{T} = 0$, i.e., \mathcal{H} is torsion-free.

Proof. Set $U = X \setminus Z$, and let $i : U \rightarrow X$ be the inclusion. Let $\varphi : \mathcal{G} \rightarrow \mathcal{H}/\mathcal{T}$ be the composite of the two quotients $\mathcal{G} \rightarrow \mathcal{H}$ and $\mathcal{H} \rightarrow \mathcal{H}/\mathcal{T}$. Let \mathcal{K} be the kernel of φ . We have the following exact sequence by the snake lemma.

$$0 \rightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{K} \rightarrow \mathcal{T} \rightarrow 0$$

Since the support of \mathcal{T} is outside U , we see that $\mathcal{F}|_U \cong \mathcal{K}|_U$. Since \mathcal{K} is a subsheaf of a torsion-free sheaf \mathcal{G} , \mathcal{K} is torsion-free. Hence the canonical morphism $\mathcal{K} \rightarrow i_*(\mathcal{K}|_U)$ is injective. On the other hand, we have isomorphisms $\mathcal{F} \cong i_*(\mathcal{F}|_U) \cong i_*(\mathcal{K}|_U)$. Therefore ψ is an isomorphism, and thus $\mathcal{T} \cong 0$. Hence \mathcal{H} is torsion-free. \square

Recall here that $\det : K(X) \rightarrow \text{Pic } X$ is defined since X is smooth and projective and thus every coherent sheaf admits a finite locally free resolution. Here $K(X)$ denotes the Grothendieck group of X .

In the rest of this section, we assume that $\text{Pic } X \cong \mathbb{Z}$, and let $\mathcal{O}_X(1)$ denote the ample generator of $\text{Pic } X$. Let d be the integer such that $\det \mathcal{E} \cong \mathcal{O}_X(d)$.

Lemma 5.5. *If \mathcal{G} is a quotient coherent sheaf of \mathcal{E} , then $\det \mathcal{G}$ is nef. Moreover if $\det \mathcal{G} \cong \mathcal{O}_X$ then the support of the torsion subsheaf of \mathcal{G} has codimension ≥ 2 in X .*

Proof. First suppose that \mathcal{G} is torsion free, and let Z be the singular locus of \mathcal{G} , i.e., the locus where \mathcal{G} is not locally free. Then Z has codimension ≥ 2 . Set $U = X \setminus Z$, and let $i : U \rightarrow X$ be the inclusion. Observe that $\det \mathcal{G}$ is equal to the sheaf $i_*(\det(\mathcal{G}|_U))$. Let s be the rank of \mathcal{G} . Then the surjection $\mathcal{E} \rightarrow \mathcal{G}$ induces a morphism $\wedge^s \mathcal{E} \rightarrow \det \mathcal{G}$ which is surjective on U . Suppose, to the contrary, that $\det \mathcal{G}$ is not nef. Then $\det \mathcal{G}$ is isomorphic to $\mathcal{O}(k)$ for some negative integer k since $\text{Pic } X \cong \mathbb{Z}$. Let C be a general smooth curve that intersect with U . Then the restriction $\wedge^s \mathcal{E}|_C \rightarrow \det \mathcal{G}|_C \cong \mathcal{O}_C(k)$ is non-zero, and the image of this morphism is a line bundle of negative degree on C . This contradicts that $\wedge^s \mathcal{E}|_C$ is nef. Therefore $\det \mathcal{G}$ is nef.

Now consider the general case. Let \mathcal{T} be the torsion subsheaf of \mathcal{G} . Then $\det(\mathcal{G}/\mathcal{T})$ is nef by the consideration above. Since $\det \mathcal{G} \cong \det \mathcal{T} \otimes \det(\mathcal{G}/\mathcal{T})$, it is enough to show that $\det \mathcal{T}$ is nef. Suppose, for a moment, that $\mathcal{T} \cong \mathcal{O}_D(u) := \mathcal{O}_X(u) \otimes \mathcal{O}_D$ for some closed subvariety D of X and an integer u . If D has codimension ≥ 2 in X , then $\det \mathcal{T} = \mathcal{O}_X$. If D has codimension one, then D is an ample Cartier divisor since $\text{Pic } X \cong \mathbb{Z}$. Thus $\det \mathcal{T}$ is isomorphic to an ample line bundle $\mathcal{O}_X(D)$. Therefore $\det \mathcal{T}$ is nef if $\mathcal{T} \cong \mathcal{O}_D(u)$. Now, for a general \mathcal{T} , recall that \mathcal{T} has an “irreducible decomposition”, i.e., a filtration every graded piece of which is of the form $\mathcal{O}_D(u)$ for some integer u where D is a closed subvariety defined by an associated point of \mathcal{T} . Since the assertion holds for every graded piece, we conclude that $\det \mathcal{T}$ is nef.

Finally if $\det \mathcal{G} \cong \mathcal{O}_X$ it follows from the consideration above that the support of \mathcal{T} has codimension ≥ 2 in X . \square

Remark 5.6. . *If $\dim X = 2$ and \mathcal{G} is torsion-free, or if $\dim X = 1$, then the assumption that $\text{Pic } X \cong \mathbb{Z}$ is unnecessary in Lemma 5.5.*

Remark 5.7. If \mathcal{G} is not locally free, then $\det \mathcal{G} \neq \wedge^s \mathcal{G}$ in general where $s = \text{rank } \mathcal{G}$. For example, if $\dim X = 1$ and $\mathcal{G} = \mathcal{O}^{\oplus 2} \oplus k(p)$ where $k(p)$ is the residue field at a point $p \in X$, then $\det \mathcal{G} \cong \mathcal{O}(p)$ whereas $\wedge^2 \mathcal{G} \cong \mathcal{O} \oplus k(p)^{\oplus 2}$. If $X = \mathbb{P}^2$ and $\mathcal{G} \cong \mathcal{O} \oplus \mathfrak{m}$ where \mathfrak{m} is the maximal ideal of a point $p \in X$, then $\det \mathcal{G} \cong \mathcal{O}$ whereas $\wedge^2 \mathcal{G} \cong \mathfrak{m}$.

Proposition 5.8. Suppose that $H^0(X, \mathcal{O}_X(1)) \neq 0$ and that $\dim \text{Hom}(\mathcal{O}(d), \mathcal{E}) = 0$. If $\dim \text{Hom}(\mathcal{O}(d-1), \mathcal{E}) \geq 2$, then $d \leq 2$. Moreover if $d = 2$ then $\dim \text{Hom}(\mathcal{O}(1), \mathcal{E}) = 2$ and we have an exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}(1), \mathcal{E}) \otimes \mathcal{O}(1) \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$$

with \mathcal{G} a vector bundle.

Proof. Let σ be a non-zero element of $\text{Hom}(\mathcal{O}(d-1), \mathcal{E})$, and s the corresponding element of $H^0(\mathcal{E}(1-d))$. Since \mathcal{E} is torsion-free, σ is generically injective. Moreover σ is injective since $\mathcal{O}(1)$ is torsion-free. Since $H^0(\mathcal{O}(1)) \neq 0$, we have an injection $\text{Hom}(\mathcal{O}(i+1), \mathcal{E}) \rightarrow \text{Hom}(\mathcal{O}(i), \mathcal{E})$ for any integer i . Since $\text{Hom}(\mathcal{O}(d), \mathcal{E}) = 0$, we have $\text{Hom}(\mathcal{O}(i), \mathcal{E}) = 0$ for all $i \geq d$. Since \mathcal{E} is locally free, this implies that the zero locus $(s)_0$ of s has codimension ≥ 2 . Define a coherent sheaf \mathcal{F} by the following exact sequence

$$0 \rightarrow \mathcal{O}(d-1) \xrightarrow{\sigma} \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0.$$

Then \mathcal{F} is locally free outside the zero locus $(s)_0$ of s . Thus the support of torsion subsheaf of \mathcal{F} is contained in $(s)_0$. Hence \mathcal{F} is torsion-free by Lemma 5.4.

We have an exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}(d-1), \mathcal{O}(d-1)) \rightarrow \text{Hom}(\mathcal{O}(d-1), \mathcal{E}) \rightarrow \text{Hom}(\mathcal{O}(d-1), \mathcal{F}).$$

In particular, we see that the image of the map $\text{Hom}(\mathcal{O}(d-1), \mathcal{E}) \rightarrow \text{Hom}(\mathcal{O}(d-1), \mathcal{F})$ has dimension ≥ 1 . Let τ be a non-zero element in the image. Since \mathcal{F} is torsion-free, τ is generically injective. Moreover τ is injective since $\mathcal{O}(d-1)$ is torsion-free. Define a coherent sheaf \mathcal{G} by the following exact sequence

$$0 \rightarrow \mathcal{O}(d-1) \xrightarrow{\tau} \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0.$$

Let V be the pull back of the one-dimensional subspace $K\tau$ generated by τ by the map $\text{Hom}(\mathcal{O}(d-1), \mathcal{E}) \rightarrow \text{Hom}(\mathcal{O}(d-1), \mathcal{F})$. Then V has dimension two. By the snake lemma, we see that there exists the following exact sequence

$$0 \rightarrow V \otimes \mathcal{O}(d-1) \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0.$$

Hence $\det \mathcal{G} \cong \mathcal{O}_X(2-d)$. Since $\det \mathcal{G}$ is nef by Lemma 5.5, we conclude that $d \leq 2$.

Suppose moreover that $d = 2$. Then $\det \mathcal{G} \cong \mathcal{O}_X$. Lemma 5.5 implies that the support of the torsion subsheaf of \mathcal{G} has codimension ≥ 2 in X . Then \mathcal{G} is torsion-free by Lemma 5.4. Next we show that $V = \text{Hom}(\mathcal{O}(1), \mathcal{E})$. Suppose, to the contrary, that $V \subsetneq \text{Hom}(\mathcal{O}(1), \mathcal{E})$. Let v be an element of $\text{Hom}(\mathcal{O}(1), \mathcal{E}) \setminus V$. Then, since \mathcal{G} and $\mathcal{O}(1)$ are torsion-free, v defines an injective morphism $\mathcal{O}_X(1) \rightarrow \mathcal{G}$, which implies that \mathcal{G} has a quotient sheaf \mathcal{H} with $\det \mathcal{H} \cong \mathcal{O}_X(-1)$. On the other hand, since \mathcal{H} is also a quotient sheaf of \mathcal{E} , $\det \mathcal{H}$ is nef by Lemma 5.5. This is a contradiction. Therefore $V = \text{Hom}(\mathcal{O}(1), \mathcal{E})$. Finally we show that \mathcal{G} is a vector bundle. Let Z be the singular locus of \mathcal{G} . Since \mathcal{G} is torsion-free, Z has codimension ≥ 2 . For any point x of X , take a curve C which contains x and is not contained in Z . Let $\tilde{C} \rightarrow C$ be the normalization. Then $\mathcal{G} \otimes \mathcal{O}_{\tilde{C}}$ is generically free of rank $r - 2$. Thus we have an exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{O}(1), \mathcal{E}) \otimes \mathcal{O}_{\tilde{C}}(1) \rightarrow \mathcal{E} \otimes \mathcal{O}_{\tilde{C}} \rightarrow \mathcal{G} \otimes \mathcal{O}_{\tilde{C}} \rightarrow 0.$$

Since $\det \operatorname{Hom}(\mathcal{O}(1), \mathcal{E}) \otimes \mathcal{O}_{\tilde{C}}(1) \cong \det \mathcal{E} \otimes \mathcal{O}_{\tilde{C}}$, we see that $\det(\mathcal{G} \otimes \mathcal{O}_{\tilde{C}}) \cong \mathcal{O}_{\tilde{C}}$. Then $\mathcal{G} \otimes \mathcal{O}_{\tilde{C}}$ is torsion-free by Lemma 5.5 and Remark 5.6. Thus $\mathcal{G} \otimes \mathcal{O}_{\tilde{C}}$ is locally free, and hence \mathcal{G} is locally free at x . Therefore \mathcal{G} is a vector bundle. \square

Corollary 5.9. *Suppose that every nef vector bundle on X with trivial determinant is isomorphic to a direct sum of copies of \mathcal{O}_X , that $H^0(X, \mathcal{O}_X(1)) \neq 0$, and that $H^1(X, \mathcal{O}(1)) = 0$. If $d = 2$, $\dim \operatorname{Hom}(\mathcal{O}(2), \mathcal{E}) = 0$, and $\dim \operatorname{Hom}(\mathcal{O}(1), \mathcal{E}) \geq 2$, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$.*

Proof. By Proposition 5.8, $\dim \operatorname{Hom}(\mathcal{O}(1), \mathcal{E}) = 2$ and we have the following exact sequence

$$0 \rightarrow \operatorname{Hom}(\mathcal{O}(1), \mathcal{E}) \otimes \mathcal{O}(1) \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$$

with \mathcal{G} a vector bundle. We see that \mathcal{G} is nef with $\det \mathcal{G} \cong \mathcal{O}_X$. Therefore $\mathcal{G} \cong \mathcal{O}^{\oplus r-2}$ by assumption. Since $H^1(X, \mathcal{O}(1)) = 0$, we conclude that $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$. \square

Remark 5.10. *The assumption on $(X, \mathcal{O}(1))$ of Corollary 5.9 is satisfied if $(X, \mathcal{O}(1))$ is either $(\mathbb{P}^n, \mathcal{O}(1))$ or $(\mathbb{Q}^n, \mathcal{O}(1))$.*

Remark 5.11. *Set $X = \mathbb{P}^2$, and set $\mathcal{F} = \mathfrak{m}_{x,X} \otimes \mathcal{O}(1)$, where $\mathfrak{m}_{x,X}$ is the ideal sheaf on X of a point x of X . Then \mathcal{F} is a torsion-free sheaf, and we see that $\dim H^0(\mathcal{F}) = 2$ and that $\dim H^0(\mathcal{F}(-1)) = 0$. Let H be a line passing through x , and let t be the corresponding element of $H^0(\mathcal{F})$. Set $U = X \setminus \{x\}$. Then H is the closure of the zero locus $(t|_U)_0$ of the restriction $t|_U$ of t to U . However the restriction $t|_H \in H^0(\mathcal{F}|_H)$ of t to H does not vanish. The element $t|_H$ generates the torsion subsheaf of $\mathcal{F}|_H$.*

6. THE CASE WHERE X IS A PROJECTIVE SPACE

Let $X, G_0, \dots, G_m, G, A, \mathcal{E}, \mathcal{O}_X(1), d_{\min}$, and $e_{l,j}$ be as in § 2 for $0 \leq j \leq l \leq m$. Assume that \mathcal{E} be a nef vector bundle of rank r as in § 3. In this section, we assume that X, G_0, \dots, G_m , and $\mathcal{O}_X(1)$ are as in the case (1) in § 3. Let d be as in § 3.

Lemma 6.1. *The following holds.*

- (1) *Let H be a hyperplane of \mathbb{P}^n . Then d_{\min} for $\mathcal{E}|_H$ with respect to $(\mathcal{O}_H, \dots, \mathcal{O}_H(n-1))$ is less than or equal to d_{\min} for \mathcal{E} with respect to $(\mathcal{O}, \dots, \mathcal{O}(n))$.*
- (2) *Suppose that $H^0(\mathcal{E}(k-3)) = 0$ and $d_{\min} \leq k$. Then, for any l -dimensional linear section \mathbb{P}^l of \mathbb{P}^n with $l \geq 2$, the restriction map $H^0(\mathcal{E}(k-2)) \rightarrow H^0(\mathcal{E}|_{\mathbb{P}^l}(k-2))$ is an isomorphism.*

Proof. (1) Set $G' = \bigoplus_{i=0}^{n-1} \mathcal{O}(i)$. For any $d' \geq d_{\min}$, we have the following distinguished triangle

$$\operatorname{RHom}(G'(1), \mathcal{E}(d')) \rightarrow \operatorname{RHom}(G', \mathcal{E}(d')) \rightarrow \operatorname{RHom}(G'|_H, \mathcal{E}|_H(d')) \rightarrow .$$

Since $\operatorname{Ext}^q(G'(1), \mathcal{E}(d')) = 0$ and $\operatorname{Ext}^q(G', \mathcal{E}(d')) = 0$ for $q > 0$, we obtain for $q > 0$ that $\operatorname{Ext}^q(G'|_H, \mathcal{E}|_H(d')) = 0$. Therefore d_{\min} for $\mathcal{E}|_H$ is less than or equal to d_{\min} for \mathcal{E} .

(2) Suppose that $n \geq 3$, and let H be a hyperplane section of \mathbb{P}^n . Since $n \geq 3$ and $d_{\min} \leq k$, we see that $H^1(\mathcal{E}(k-3)) = 0$. Since $H^0(\mathcal{E}(k-3)) = 0$ by assumption, we obtain $H^0(\mathcal{E}(k-2)) \cong H^0(\mathcal{E}|_H(k-2))$. Since the statement (1) holds, we now obtain the statement by induction. \square

Lemma 6.2. *Suppose that $\operatorname{Hom}(\mathcal{O}(d), \mathcal{E}) = 0$. If there exists a non-zero element s of $H^0(\mathcal{E}(1-d))$, then the zero locus $(s)_0$ of s is either empty or a (reduced) point. Moreover if $(s)_0$ is a point, then $r \geq n \geq 2$.*

Proof. Set $Z = (s)_0$. Since $H^0(\mathcal{E}(-d)) = 0$, Z has codimension $c \geq 2$ in \mathbb{P}^n . Suppose that Z is not empty.

We show that the length of the non-empty intersection $Z \cap L$ of Z and a line L is one unless $Z \cap L = L$. Let L be a line such that $Z \cap L$ is a non-empty finite set, and let l be the length of $Z \cap L$; l is a positive integer. Then we have $\mathcal{O}_L(l)$ as a subbundle of $\mathcal{E}|_L(1-d)$. Thus $\mathcal{E}|_L$ has $\mathcal{O}_L(l+d-1)$ as a direct summand. On the other hand, since \mathcal{E} is nef, the degree of a direct summand of \mathcal{E} is at most d . Thus $l = 1$. This implies that Z in \mathbb{P}^n has no secant lines that is not contained in Z , and hence we see that Z is a linear subspace \mathbb{P}^{n-c} as sets. Moreover we see that Z is reduced. Indeed, let p be a point of Z and let I be the ideal sheaf of $Z \cap \mathbb{A}^n$ in a linear affine open subset \mathbb{A}^n containing p . Let (x_1, \dots, x_n) be the affine coordinates of \mathbb{A}^n . We may assume that $p = (0, \dots, 0)$ and that the radical \sqrt{I} of I is (x_1, \dots, x_c) . Let l be the minimal integer such that $x_1^l \in I$, and let L be a line in \mathbb{P}^n defined as the closure of the affine line defined by (x_2, \dots, x_n) . Then $Z \cap L$ is a non-empty finite subscheme of length at least l . Thus we see, by the same argument as above, that $l = 1$. Hence $x_1 \in I$. By the same way, we see that $x_i \in I$ for all $1 \leq i \leq c$. Therefore we conclude that Z is reduced and thus Z is a linear subscheme \mathbb{P}^{n-c} of \mathbb{P}^n .

Let \mathcal{I} be the ideal sheaf of Z in \mathbb{P}^n . Then the conormal bundle $\mathcal{I}/\mathcal{I}^2$ is isomorphic to $\mathcal{O}_Z(-1)^{\oplus c}$. On the other hand, we have a surjection $\mathcal{E}^\vee(d-1) \rightarrow \mathcal{I}$. Suppose that Z contains a line L_0 . Then we have a surjection

$$\mathcal{E}^\vee|_{L_0}(d-1) \rightarrow \mathcal{O}_{L_0}(-1)^{\oplus c}.$$

In particular, $\mathcal{E}^\vee|_{L_0}(d-1)$ has an $\mathcal{O}_{L_0}(-1)$ as a quotient. This implies that $\mathcal{E}^\vee|_{L_0}(d-1)$ is isomorphic to $\mathcal{O}_{L_0}(-1) \oplus \mathcal{O}_{L_0}(d-1)^{\oplus r-1}$ since \mathcal{E} is nef. However this means that $\mathcal{E}^\vee|_{L_0}(d-1)$ cannot have $\mathcal{O}_{L_0}(-1)^{\oplus c}$ as a quotient since $d \geq 1$ and $c \geq 2$. This is a contradiction. Therefore Z is a point. \square

In the rest of this section, we assume that the base field K is of characteristic zero. Theorem 6.3 below is a part of [16, Theorem 1], and is also a consequence of [4, IV-2.2 Proposition]. We give a different proof of this result based on our framework: general restrictions on $e_{i,j}$'s and d_{\min} obtained so far enable us to prove this theorem immediately.

Theorem 6.3. *Suppose that $d = 1$, i.e., that $\det \mathcal{E} \cong \mathcal{O}(1)$. Then \mathcal{E} is isomorphic to either $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1}$ or $T_{\mathbb{P}^n}(-1) \oplus \mathcal{O}^{\oplus r-n}$.*

Proof. We see first that $d_{\min} \leq 1$ by Corollary 4.2. If $r = 1$, then $\mathcal{E} \cong \mathcal{O}(1)$. Suppose that $r \geq 2$. Then $d_{\min} \geq 0$ by Proposition 3.1 (1) (b). Suppose that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) \neq 0$. Then $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1}$ by Proposition 5.2. Suppose that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$. If d_{\min} were zero, the standard resolution of \mathcal{E} with respect to $(\mathcal{O}, \dots, \mathcal{O}(n))$ implies that $\mathcal{E} \cong \mathcal{O}^{\oplus r}$, which contradicts that $d = 1$. Therefore $d_{\min} = 1$. Then the standard resolution of $\mathcal{E}(1)$ modified by Proposition 2.7 is

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus r+1} \rightarrow \mathcal{E}(1) \rightarrow 0,$$

since $\det \mathcal{E}(1) \cong \mathcal{O}(r+1)$. Then we see $r \geq n$ and $\mathcal{E} \cong T_{\mathbb{P}^n}(-1) \oplus \mathcal{O}^{\oplus r-n}$, because \mathcal{E} is a vector bundle. \square

Theorem 6.4. *Suppose that $\text{Hom}(\mathcal{O}(d), \mathcal{E}) = 0$ and that $\text{Hom}(\mathcal{O}(d-1), \mathcal{E}) \neq 0$. Then \mathcal{E} satisfies one of the following:*

- (1) $\mathcal{E} \cong \mathcal{O}(d-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-2}$.

(2) \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}(d-1) \oplus \mathcal{O}^{\oplus r} \rightarrow \mathcal{E} \rightarrow 0.$$

Proof. Let s be a non-zero element of $H^0(\mathcal{E}(1-d))$. Let

$$0 \rightarrow \mathcal{O}(d-1) \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

be an exact sequence of coherent sheaves defined by s .

If $(s)_0 = \emptyset$, then \mathcal{F} is a nef vector bundle with $\det \mathcal{F} \cong \mathcal{O}(1)$. Theorem 6.3 then implies that \mathcal{E} satisfies (1) or (2) of the theorem.

If $(s)_0 \neq \emptyset$, then $(s)_0$ is a point z by Lemma 6.2. Consider the projection from the point z . By eliminating the indeterminacy, we get a morphism $f : Y \rightarrow \mathbb{P}^{n-1}$ where $\varphi : Y \rightarrow \mathbb{P}^n$ is the blowing-up at the point z . Let E be the exceptional divisor of φ . We see that f is a \mathbb{P}^1 -bundle. Then we get the following exact sequence

$$0 \rightarrow \varphi^* \mathcal{O}(d-1) \otimes \mathcal{O}(E) \rightarrow \varphi^* \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$$

for a vector bundle \mathcal{G} on Y . We see that $\mathcal{G}|_F \cong \mathcal{O}_F^{\oplus r-1}$ for any fiber $F \cong \mathbb{P}^1$ of f . Thus there exists a vector bundle \mathcal{H} on \mathbb{P}^{n-1} such that $\mathcal{G} \cong f^* \mathcal{H}$. By restricting the exact sequence

$$0 \rightarrow \varphi^* \mathcal{O}(d-1) \otimes \mathcal{O}(E) \rightarrow \varphi^* \mathcal{E} \rightarrow f^* \mathcal{H} \rightarrow 0$$

to the exceptional divisor E , we see that \mathcal{H} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus r} \rightarrow \mathcal{H} \rightarrow 0.$$

Since \mathcal{H} is a vector bundle on \mathbb{P}^{n-1} , we infer that $r \geq n$ and that

$$\mathcal{H} \cong T_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}^{\oplus r-n}.$$

Hence $\varphi^* \mathcal{E}$ has $\varphi^* \mathcal{O}^{\oplus r-n}$ as a direct summand, and thus \mathcal{E} has $\mathcal{O}^{\oplus r-n}$ as a direct summand. Therefore we have

$$\mathcal{E} \cong \mathcal{O}^{\oplus r-n} \oplus \mathcal{E}_0$$

for some nef vector bundle \mathcal{E}_0 of rank n with $\det \mathcal{E}_0 \cong \mathcal{O}(d)$. We may assume that $d \geq 2$ and that s is a non-zero element of $H^0(\mathcal{E}_0(1-d))$. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}(d-1) \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F}_0 \rightarrow 0,$$

where \mathcal{F}_0 is a torsion-free coherent sheaf with $\det \mathcal{F}_0 \cong \mathcal{O}(1)$.

We claim here that $\mathbb{P}(\mathcal{F}_0)$ is nonsingular, although \mathcal{F}_0 is not a vector bundle. Let $\pi : \mathbb{P}(\mathcal{E}_0) \rightarrow \mathbb{P}^n$ be the projection. It is clear that $\mathbb{P}(\mathcal{F}_0) \cap \pi^{-1}(\mathbb{P}^n \setminus \{z\})$ is nonsingular. We may assume that, locally around z , the section s of $\mathcal{E}_0(1-d)$ can be written as

$$s = z_1 e_1 + z_2 e_2 + \cdots + z_n e_n,$$

where (z_1, \dots, z_n) is a local coordinate system around z with $z = (0, \dots, 0)$ and (e_1, \dots, e_n) is a locally free basis of $\mathcal{E}_0(1-d)$ around z . Regarding $(e_1; \dots; e_n)$ as a homogeneous coordinate system on $\pi^{-1}(z) \cong \mathbb{P}^{n-1}$, we see that $ds = dz_1 e_1 + dz_2 e_2 + \cdots + dz_n e_n$ on the cover of the fiber $\pi^{-1}(z)$ and that ds does not vanish on the fiber $\pi^{-1}(z)$. Therefore we conclude that $\mathbb{P}(\mathcal{F}_0)$ is also nonsingular along the fiber $\pi^{-1}(z)$.

Now the Kodaira vanishing theorem implies that d_{\min} for \mathcal{F}_0 is less than two by the similar argument as in the proof of Lemma 4.1 (1).

If $H^0(\mathcal{F}_0(-1)) \neq 0$, then the similar argument as in the proof of Proposition 5.8 and Corollary 5.9 implies that $\mathcal{E}_0 \cong \mathcal{O}(d-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{\oplus n-2}$. Thus we obtain the case (1) of the theorem.

Suppose that $H^0(\mathcal{F}_0(-1)) = 0$. We claim here that d_{\min} for \mathcal{F}_0 is one. Indeed, if $\text{rank } \mathcal{F}_0 = 1$, then we see that $\mathcal{F}_0 \cong \mathfrak{m}_z(1)$, where \mathfrak{m}_z is the ideal sheaf of z , and the claim follows. If $\text{rank } \mathcal{F}_0 \geq 2$, then we first infer that d_{\min} for \mathcal{F}_0 is greater than or equal to zero by the similar argument as in the proof of Proposition 3.1, since \mathcal{F}_0 is a torsion-free quotient of a nef vector bundle \mathcal{E}_0 . If d_{\min} for \mathcal{F}_0 were zero, then the standard resolution of \mathcal{F}_0 shows that \mathcal{F}_0 would be isomorphic to $\mathcal{O}^{\oplus n-1}$, which contradicts the fact that $\det \mathcal{F}_0 \cong \mathcal{O}(1)$. Therefore we conclude that the claim holds. Then the standard resolution of $\mathcal{F}_0(1)$ modified according to Proposition 2.7 is

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus n} \rightarrow \mathcal{F}_0(1) \rightarrow 0,$$

which implies that \mathcal{E} is in the case (2) of the theorem. \square

The following is the main part of [16, Theorem 1] of Peternell-Szurek-Wisniewski. Based on our framework, we give a different proof of this result. See Remark 6.6 for the seeming difference of Theorem 6.5 and [16, Theorem 1].

Theorem 6.5. *Suppose that $d = 2$, i.e., that $\det \mathcal{E} \cong \mathcal{O}(2)$. Then \mathcal{E} satisfies one of the following:*

- (1) $\mathcal{E} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-1}$.
- (2) $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}$.
- (3) \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r} \rightarrow \mathcal{E} \rightarrow 0.$$

- (4) \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1)^{\oplus 2} \rightarrow \mathcal{O}^{\oplus r+2} \rightarrow \mathcal{E} \rightarrow 0.$$

- (5) \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow 0.$$

- (6) $n = 3$ and \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 4} \rightarrow \mathcal{O}^{\oplus r+3} \rightarrow \mathcal{E} \rightarrow 0.$$

Proof. Suppose that $\text{Hom}(\mathcal{O}(2), \mathcal{E}) \neq 0$. Then we see that $\mathcal{E} \cong \mathcal{O}(2) \oplus \mathcal{O}^{\oplus r-1}$ by Proposition 5.2. In the following, we assume that $\text{Hom}(\mathcal{O}(2), \mathcal{E}) = 0$. If $\dim \text{Hom}(\mathcal{O}(1), \mathcal{E}) \neq 0$, then Theorem 6.4 shows that \mathcal{E} is either in the case (2) or in the case (3) of the theorem. We assume that $\dim \text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$ in the following. If $d_{\min} \leq 0$, then the standard resolution of \mathcal{E} implies that $\mathcal{E} \cong \mathcal{O}^{\oplus r}$, which contradicts the assumption $d = 2$. Thus $d_{\min} \geq 1$. Then we have $n \geq 2$. We also see $d_{\min} \leq 2$ by Corollary 4.2.

Suppose that $d_{\min} = 1$. Then the standard resolution of $\mathcal{E}(1)$ modified according to Proposition 2.7 together with $\mathcal{O}(-1)$ -twist is

$$0 \rightarrow \mathcal{O}(-1)^{\oplus e_{1,0}-e_{0,0}} \rightarrow \mathcal{O}^{\oplus e_{0,1}} \rightarrow \mathcal{E} \rightarrow 0.$$

Since $d = 2$, we have $e_{1,0} - e_{0,0} = 2$, and thus $e_{0,1} = r + 2$. This is the case (4) of the theorem.

In the following, we assume that $d_{\min} = 2$. We shall apply to $\mathcal{E}(1)$ the Bondal spectral sequence [13, Theorem 1]

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}(1)), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E}(1) & \text{if } p+q = 0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

So we first claim that, under the assumption that d_{\min} for \mathcal{E} is two, equality

$$H^q(\mathcal{E}(1-n)) = \begin{cases} K & \text{if } q = n-1 \\ 0 & \text{if } q > 0 \text{ and } q \neq n-1 \end{cases}$$

holds unless $n = 3$ and $H^1(\mathcal{E}(-2)) \cong K$.

We shall prove this claim by induction on n unless $n = 3$ and $H^1(\mathcal{E}(-2)) \cong K$. Let H be a hyperplane in \mathbb{P}^n . We have an exact sequence

$$0 \rightarrow \mathcal{E}(1-n) \rightarrow \mathcal{E}(2-n) \rightarrow \mathcal{E}|_H(1-(n-1)) \rightarrow 0.$$

Since $d_{\min} = 2$, we see that $H^q(\mathcal{E}(2-n)) = 0$ for all $q > 0$ and that $H^q(\mathcal{E}(1-n)) \neq 0$ for some $q > 0$.

Suppose that $n = 2$. The Riemann-Roch formula for a vector bundle \mathcal{E} of rank r on \mathbb{P}^2 is

$$\chi(\mathcal{E}) = r + \frac{1}{2}d(d+3) - c_2(\mathcal{E}).$$

Since $d = 2$ by assumption, the above formula implies that $h^0(\mathcal{E}) = \chi(\mathcal{E}) = r + 5 - c_2(\mathcal{E})$. Since $0 \leq H(\mathcal{E})^{r+1} = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$, we have $c_2(\mathcal{E}) \leq 4$. Hence $h^0(\mathcal{E}) \geq r + 1$. Since $h^0(\mathcal{E}|_H) = r + 2$, $h^0(\mathcal{E}(-1)) = 0$, and $h^q(\mathcal{E}|_H) = 0$ for all $q > 0$, we see that $h^q(\mathcal{E}(-1)) = 0$ for all $q \geq 2$, that $h^1(\mathcal{E}(-1)) = 1$, and that $h^0(\mathcal{E}) = r + 1$. Hence the claim holds for $n = 2$.

Suppose that $n \geq 3$ and that the claim holds for $n-1$. Since we have $H^q(\mathcal{E}(2-n)) = 0$ for all $q \geq 0$, we see that

$$H^q(\mathcal{E}(1-n)) \cong H^{q-1}(\mathcal{E}|_H(1-(n-1))) \text{ for all } q \geq 1.$$

The point here is to show that d_{\min} for $\mathcal{E}|_H$ is two, unless $n = 3$ and $H^1(\mathcal{E}(-2)) \cong K$. Note first that d_{\min} for $\mathcal{E}|_H$ is less than or equal to two by Lemma 6.1 (1) and that we have proved the theorem in case $d_{\min} \leq 1$. Now suppose that d_{\min} for $\mathcal{E}|_H$ is less than one. Then $\mathcal{E}|_H$ splits, so that \mathcal{E} also splits by [14, Chap. 1, Theorem 2.3.2]. This contradicts that $d_{\min} = 2$. Suppose that d_{\min} for $\mathcal{E}|_H$ is one. Then $\mathcal{E}|_H$ does not split. We have $H^{q-1}(\mathcal{E}|_H(1-(n-1))) = 0$ for all $q \geq 2$. Hence $H^q(\mathcal{E}(1-n)) = 0$ for all $q \geq 2$. Since $H^q(\mathcal{E}(1-n)) \neq 0$ for some $q \geq 1$, we see that $H^1(\mathcal{E}(1-n)) \neq 0$. Thus $H^0(\mathcal{E}|_H(1-(n-1))) \neq 0$. Hence $n = 3$ since $\mathcal{E}|_H$ does not split. Then $\mathcal{E}|_H$ is in the case (3) of the theorem, and thus $H^0(\mathcal{E}|_H(-1)) \cong K$. Hence $H^1(\mathcal{E}(-2)) \cong K$. This shows that d_{\min} for $\mathcal{E}|_H$ is two unless $n = 3$ and $H^1(\mathcal{E}(-2)) \cong K$. Now the claim holds by induction, because $H^0(\mathcal{E}|_H(1-(n-1))) = 0$ if d_{\min} for $\mathcal{E}|_H$ is two.

We shall show that \mathcal{E} is in the case (5) of the theorem unless $n = 3$ and $H^1(\mathcal{E}(-2)) \cong K$. First, by the claim above and the assumption that $h^0(\mathcal{E}(-1)) = 0$, we see that

$$\text{Ext}^q(G, \mathcal{E}(1)) = \begin{cases} \text{Hom}(\mathcal{O}, \mathcal{E}(1)) \oplus \text{Hom}(\mathcal{O}(1), \mathcal{E}(1)) & \text{if } q = 0 \\ \text{Ext}^{n-1}(G_n, \mathcal{E}(1)) = K & \text{if } q = n-1 \\ 0 & \text{if } q > 0 \text{ and } q \neq n-1. \end{cases}$$

Hence we have $E_2^{p,q} = 0$ unless $q = n-1$ or 0 . As we have shown in the proof of [13, Proposition 1], we also infer that

$$E_2^{p,n-1} = \mathcal{H}^p(\text{Ext}^{n-1}(G, \mathcal{E}(1)) \otimes_{\mathbb{L}_A} G) = \begin{cases} \mathcal{O}(-1) & \text{if } p = -n \\ 0 & \text{if } p \neq -n. \end{cases}$$

We finally see that a right A -module $\text{Hom}(G, \mathcal{E}(1))$ has a projective resolution of the following form

$$0 \rightarrow P_0^{\oplus f_{1,0}} \rightarrow P_0^{\oplus f_{0,0}} \oplus P_1^{\oplus f_{0,1}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0,$$

where P_0 and P_1 are as in § 2, $f_{0,j} = \dim \text{Hom}(G_j, \mathcal{E}(1))$ ($j = 0, 1$), and $f_{1,0} = (n+1)f_{0,1}$. Hence we see that

$$E_2^{p,0} = \begin{cases} \text{Ker}(\mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}}) & \text{if } p = -1 \\ \text{Coker}(\mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}}) & \text{if } p = 0 \\ 0 & \text{if } p \neq -1, 0. \end{cases}$$

Thus we infer that

$$E_\infty^{p,q} = \begin{cases} E_n^{-n,n-1} = \text{Ker}(\mathcal{O}(-1) \rightarrow E_2^{0,0}) & \text{if } (p,q) = (-n, n-1) \\ E_n^{0,0} = \text{Coker}(\mathcal{O}(-1) \rightarrow E_2^{0,0}) & \text{if } (p,q) = (0,0) \\ E_2^{-1,0} & \text{if } (p,q) = (-1,0) \\ 0 & \text{otherwise.} \end{cases}$$

The Bondal spectral sequence then shows that

$$E_\infty^{p,q} = \begin{cases} \mathcal{E}(1) & \text{if } (p,q) = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

This shows that we have exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-1) \rightarrow E_2^{0,0} \rightarrow \mathcal{E}(1) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow E_2^{0,0} \rightarrow 0. \end{aligned}$$

Therefore we get an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since $\det \mathcal{E}(1) \cong \mathcal{O}(r+2)$, we have $f_{0,1} = r+1$. We claim here that the composite of the inclusion $\mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus f_{1,0}}$, the morphism $\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}}$, and the projection $\mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow \mathcal{O}^{\oplus f_{0,0}}$ is surjective. Assume, to the contrary, that the composite is not surjective. Then there exists a surjection $\mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow \mathcal{O}$ such that the composite $\mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow \mathcal{O}$ is zero. The morphism $\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow \mathcal{O}$ then induces a morphism $\mathcal{O}(-1) \rightarrow \mathcal{O}$, whose quotient is either $\mathcal{O}_{\mathbb{P}^n}$ or \mathcal{O}_H for some hyperplane H in \mathbb{P}^n . This implies that $\mathcal{E}(1)$ have $\mathcal{O}_{\mathbb{P}^n}$ or \mathcal{O}_H as a quotient, which contradicts that \mathcal{E} is nef. Therefore the claim holds, and we can modify the sequence above to

$$0 \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus f_{1,0}-f_{0,0}} \rightarrow \mathcal{O}(1)^{\oplus r+1} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

By looking at ranks, we infer that $f_{1,0} - f_{0,0} = 0$, and we get the case (5) of the theorem.

Finally suppose that $n = 3$ and that $H^1(\mathcal{E}(-2)) \cong K$. Then as we have seen above, we may assume that d_{\min} for $\mathcal{E}|_H$ is one. We have a distinguished triangle

$$\text{RHom}(G, \mathcal{E}(1)) \rightarrow \text{RHom}(G, \mathcal{E}(2)) \rightarrow \text{RHom}(\oplus_{i=-1}^{n-1} \mathcal{O}_H(i), \mathcal{E}|_H(1)) \rightarrow .$$

Hence we see that $\text{Ext}^q(G, \mathcal{E}(1)) = 0$ for all $q \geq 2$. Since $\text{Ext}^1(G, \mathcal{E}(2)) = 0$, we have $\text{Ext}^1(G, \mathcal{E}(1)) = \text{Ext}^1(G_n, \mathcal{E}(1)) \cong K$. Therefore we have

$$\text{Ext}^q(G, \mathcal{E}(1)) = \begin{cases} \text{Hom}(\mathcal{O}, \mathcal{E}(1)) \oplus \text{Hom}(\mathcal{O}(1), \mathcal{E}(1)) & \text{if } q = 0 \\ \text{Ext}^1(G_n, \mathcal{E}(1)) \cong K & \text{if } q = 1 \\ 0 & \text{if } q \geq 2. \end{cases}$$

Hence we have $E_2^{p,q} = 0$ for all $q \geq 2$. By the same argument as in the proof of [13, Proposition 1], we also infer that

$$E_2^{p,1} = \mathcal{H}^p(\text{Ext}^1(G, \mathcal{E}(1)) \otimes_{\mathbb{L}_A} G) = \begin{cases} \mathcal{O}(-1) & \text{if } p = -3 \\ 0 & \text{if } p \neq -3. \end{cases}$$

Finally a right A -module $\text{Hom}(G, \mathcal{E}(1))$ has a projective resolution of the following form

$$0 \rightarrow P_0^{\oplus f_{1,0}} \rightarrow P_0^{\oplus f_{0,0}} \oplus P_1^{\oplus f_{0,1}} \rightarrow \text{Hom}(G, \mathcal{E}(1)) \rightarrow 0,$$

where P_0 and P_1 are as in § 2, $f_{0,j} = \dim \text{Hom}(G_j, \mathcal{E}(1))$ ($j = 0, 1$), and $f_{1,0} = 4f_{0,1}$. Hence we see that

$$E_2^{p,0} = \begin{cases} \text{Ker}(\mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}}) & \text{if } p = -1 \\ \text{Coker}(\mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}}) & \text{if } p = 0 \\ 0 & \text{if } p \neq -1, 0. \end{cases}$$

Thus we infer that

$$E_{\infty}^{p,q} = \begin{cases} E_3^{-3,1} = \text{Ker}(\mathcal{O}(-1) \rightarrow E_2^{-1,0}) & \text{if } (p, q) = (-3, 1) \\ E_3^{-1,0} = \text{Coker}(\mathcal{O}(-1) \rightarrow E_2^{-1,0}) & \text{if } (p, q) = (-1, 0) \\ E_2^{0,0} & \text{if } (p, q) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

The Bondal spectral sequence then shows that

$$E_{\infty}^{p,q} = \begin{cases} \mathcal{E}(1) & \text{if } (p, q) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{O}(-1) \cong E_2^{-1,0}$ and we get an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus f_{1,0}} \rightarrow \mathcal{O}^{\oplus f_{0,0}} \oplus \mathcal{O}(1)^{\oplus f_{0,1}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since $\det \mathcal{E}(1) \cong \mathcal{O}(r+2)$, we see that $f_{0,1} = r+3$. By looking at ranks, we also see that $f_{1,0} - f_{0,0} = 4$. Since $\mathcal{E}(1)$ does not admit \mathcal{O} as a quotient, the sequence above can be replaced by the following one

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 4} \rightarrow \mathcal{O}(1)^{\oplus r+3} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

This is the case (6) of the theorem. □

Remark 6.6. Note that $\Omega_{\mathbb{P}^3}(2)$ has the following locally free resolution

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 4} \rightarrow \mathcal{O}^{\oplus 6} \rightarrow \Omega_{\mathbb{P}^3}(2) \rightarrow 0.$$

Thus if \mathcal{E} on \mathbb{P}^3 fits in the resolution $0 \rightarrow \mathcal{O} \rightarrow \Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}^{\oplus r-2} \rightarrow \mathcal{E} \rightarrow 0$ given in [16, Theorem 1 (2)], then \mathcal{E} also fits in a resolution

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O}(-1)^{\oplus 4} \oplus \mathcal{O} \rightarrow \mathcal{O}^{\oplus r+4} \rightarrow \mathcal{E} \rightarrow 0.$$

This implies that \mathcal{E} fits in the resolution in the case (6) of Theorem 6.5. Similarly, if \mathcal{E} fits in the resolution given in [16, Theorem 1 (3)], then it also fits in the resolution in the case (3) of Theorem 6.5. See also [13, §4, Proposition 1 and Remark 2].

Remark 6.7. Suppose that \mathcal{E} is in the case (6) of Theorem 6.5. Since $H^1(\mathcal{E}(-2)) \cong K$, \mathcal{E} cannot split. If $r \geq 3$, then $h^0(\mathcal{E}^\vee) \geq r-3$ since $h^0(\Omega_{\mathbb{P}^3}(2)) = 6$. Hence $\mathcal{E} \cong \mathcal{O}^{\oplus r-3} \oplus \mathcal{E}_0$ for some vector bundle \mathcal{E}_0 of rank three. Note that \mathcal{E}_0 is also in the case (6) of Theorem 6.5. Let \mathcal{E}_0 be a nef vector bundle in the case (6) of Theorem 6.5 and suppose that $\text{rank } \mathcal{E}_0 = 3$. Then $c_3(\mathcal{E}_0) = 0$, and we see that \mathcal{E}_0 fits in an exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0,$$

where \mathcal{F} is a nef vector bundle in the case (6) of Theorem 6.5. Let Z be the zero locus of a general element s in $H^0(\mathcal{F})$. Then Z is a smooth curve of degree two, and we have an exact sequence

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{F}(-2) \rightarrow \mathcal{I}_Z \rightarrow 0,$$

where \mathcal{I}_Z is the ideal sheaf of Z in \mathbb{P}^3 . Since $H^1(\mathcal{F}(-2)) \cong K$, we have $H^1(\mathcal{I}_Z) \cong K$. Hence Z cannot be connected. Therefore Z is a disjoint union of two lines. Since $\text{Ext}^1(\mathcal{I}_Z, \mathcal{O}(-2)) \cong K$, we conclude that $\mathcal{F} \cong \mathcal{N}(1)$, where \mathcal{N} is a null correlation bundle. Since $\text{Ext}^1(\mathcal{F}, \mathcal{O}) \cong K$, we see that \mathcal{E}_0 is either $\Omega_{\mathbb{P}^3}(2)$ or $\mathcal{O} \oplus \mathcal{N}(1)$.

7. THE CASE WHERE X IS A SMOOTH QUADRIC SURFACE

Let $X, G_0, \dots, G_m, G, A, \mathcal{E}, \mathcal{O}_X(1), d_{\min}$, and $e_{l,j}$ be as in § 2 for $0 \leq j \leq l \leq m$. Assume that \mathcal{E} be a nef vector bundle of rank r as in § 3. In this section, we assume that the base field K is of characteristic zero and that X, G_0, \dots, G_m , and $\mathcal{O}_X(1)$ are as in the case (3) in § 3 with $n = 2$. In particular, X is a smooth quadric surface \mathbb{Q}^2 , $m = 3$, and G_0, G_1, G_2, G_3 are respectively $\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)$. Let (a, b) be as in § 3.

In Theorem 7.1 below, we classify the above \mathcal{E} 's with $\det \mathcal{E} \cong \mathcal{O}(1, 1)$. Note that such \mathcal{E} 's were already classified in [21, §3] and [16, §2 Lemmas 1 and 2] (see also Remark 7.2). We give a different proof of this result in our framework.

Theorem 7.1. Suppose that $(a, b) = (1, 1)$, i.e., that $\det \mathcal{E} \cong \mathcal{O}(1, 1) = \mathcal{O}(1)$. Then \mathcal{E} satisfies one of the following:

- (1) $\mathcal{E} \cong \mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1)$.
- (2) $\mathcal{E} \cong \mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$.
- (3) \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow 0.$$

Proof. If $r = 1$, then $\mathcal{E} \cong \mathcal{O}(1)$ and $d_{\min} = -1$. We assume that $r \geq 2$ in the following. Then $0 \leq d_{\min} \leq 1$ by Proposition 3.1 (2) (e) and Corollary 4.6.

Since \mathcal{E} is nef and $\det \mathcal{E} \cong \mathcal{O}(1)$, we see that $\mathcal{E}|_L \cong \mathcal{O}_L^{\oplus r-1} \oplus \mathcal{O}_L(1)$ for any line L in \mathbb{Q}^2 .

If $\text{Hom}(\mathcal{O}(1), \mathcal{E}) \neq 0$, then it follows from Proposition 5.2 that $\mathcal{E} \cong \mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1)$. In the following we assume that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$.

Suppose that $\text{Hom}(\mathcal{O}(0, 1), \mathcal{E}) \neq 0$. Let φ be a non-zero element of $\text{Hom}(\mathcal{O}(0, 1), \mathcal{E})$. Since $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$, $\varphi|_L \neq 0$ for any line L of type $(1, 0)$ in \mathbb{Q}^2 . Hence $\varphi|_L$ makes $\mathcal{O}(0, 1)|_L$ a subbundle of $\mathcal{E}|_L$. Therefore $\mathcal{O}(0, 1)$ is a subbundle of \mathcal{E} via φ . Set $\mathcal{F} = \mathcal{E}/\mathcal{O}(0, 1)$. Then \mathcal{F} is a nef vector bundle of rank $r - 1$ with $\det \mathcal{F} \cong \mathcal{O}(1, 0)$, and thus \mathcal{F} is isomorphic to $\mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(1, 0)$. Therefore $\mathcal{E} \cong \mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$. Similarly if $\text{Hom}(\mathcal{O}(1, 0), \mathcal{E}) \neq 0$ then $\mathcal{E} \cong \mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$.

In the following, we assume that $\text{Hom}(\mathcal{O}(0,1), \mathcal{E}) = 0$ and that $\text{Hom}(\mathcal{O}(1,0), \mathcal{E}) = 0$. Under these assumptions, we have $d_{\min} = 1$. Indeed, if d_{\min} were zero, then $e_{0,3} = 0$ by the assumption that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$, and similarly $e_{0,2} = 0$ and $e_{0,1} = 0$ by the assumptions above. The standard resolution then forces \mathcal{E} to be isomorphic to $\mathcal{O}^{\oplus r}$, which contradicts the assumption that $(a, b) = (1, 1)$. Therefore $d_{\min} = 1$.

We shall apply to \mathcal{E} the Bondal spectral sequence [13, Theorem 1]

$$E_2^{p,q} = \mathcal{T}or_{-p}^A(\text{Ext}^q(G, \mathcal{E}), G) \Rightarrow E^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q=0 \\ 0 & \text{if } p+q \neq 0. \end{cases}$$

First note that $\text{Hom}(G, \mathcal{E}) \cong \text{Hom}(\mathcal{O}, \mathcal{E}) \cong H^0(\mathcal{E})$. The Riemann-Roch formula for a vector bundle \mathcal{E} of rank r on \mathbb{Q}^2 is

$$\chi(\mathcal{E}) = c'_1(\mathcal{E})c''_1(\mathcal{E}) - c_2(\mathcal{E}) + c'_1(\mathcal{E}) + c''_1(\mathcal{E}) + r,$$

where $c_1(\mathcal{E}) = (c'_1(\mathcal{E}), c''_1(\mathcal{E}))$. Since $(c'_1(\mathcal{E}), c''_1(\mathcal{E})) = (1, 1)$ by assumption, the above formula implies that $h^0(\mathcal{E}) = \chi(\mathcal{E}) = r + 3 - c_2(\mathcal{E})$. Note here that $0 \leq H(\mathcal{E})^{r+1} = c_1(\mathcal{E})^2 - c_2(\mathcal{E})$. Hence we have $c_2(\mathcal{E}) \leq 2$, and consequently $h^0(\mathcal{E}) \geq r + 1$. We have an exact sequence

$$0 \rightarrow \mathcal{E}(-1, 0) \rightarrow \mathcal{E} \rightarrow \mathcal{E}|_L \rightarrow 0,$$

where L is a line on \mathbb{Q}^2 of type $(1, 0)$. Thus we have an exact sequence

$$0 \rightarrow H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}|_L) \rightarrow H^1(\mathcal{E}(-1, 0)) \rightarrow 0$$

by our assumption. Since $h^0(\mathcal{E}|_L) = r + 1$, we infer that $h^0(\mathcal{E}) = r + 1$ and that $h^1(\mathcal{E}(-1, 0)) = 0$. Moreover we see that $\text{RHom}(\mathcal{O}(1, 0), \mathcal{E}) \cong 0$, and similarly we have $\text{RHom}(\mathcal{O}(0, 1), \mathcal{E}) \cong 0$. We have an exact sequence

$$0 \rightarrow \mathcal{E}(-1, -1) \rightarrow \mathcal{E}(0, -1) \rightarrow \mathcal{O}_L \oplus \mathcal{O}_L(-1)^{\oplus r-1} \rightarrow 0,$$

and we see that $\text{RHom}(\mathcal{O}(1, 1), \mathcal{E}) \cong K[-1]$.

Summing up, we have

$$\text{Ext}^q(G, \mathcal{E}) = \begin{cases} \text{Hom}(\mathcal{O}, \mathcal{E}) & \text{if } q = 0 \\ \text{Ext}^1(G_3, \mathcal{E}) \cong K & \text{if } q = 1 \\ 0 & \text{if } q = 2. \end{cases}$$

Hence we have $E_2^{p,q} = 0$ for all $q \geq 2$. Let S_k ($0 \leq k \leq 3$) be the right A -module corresponding to the representation such that $\text{Gr}^j S_k = 0$ for any $j \neq k$, $\text{Gr}^k S_k = K$, and all the arrows are zero. Then the right A -module $\text{Ext}^1(G, \mathcal{E})$ is isomorphic to S_3 . Note here that

$$S_3 \otimes_A^{\mathbb{L}} G \cong \mathcal{O}(-1)[2],$$

since $\text{RHom}(G, \mathcal{O}(-1)[2]) \cong \text{Ext}^2(G_3, \mathcal{O}(-1)) \cong S_3$. Hence we infer that

$$\text{Ext}^1(G, \mathcal{E}) \otimes_A^{\mathbb{L}} G \cong \mathcal{O}(-1)[2].$$

Therefore we see that

$$E_2^{p,1} = \mathcal{H}^p(\text{Ext}^1(G, \mathcal{E}) \otimes_A^{\mathbb{L}} G) = \begin{cases} \mathcal{O}(-1) & \text{if } p = -2 \\ 0 & \text{if } p \neq -2. \end{cases}$$

Finally a right A -module $\text{Hom}(G, \mathcal{E})$ is isomorphic to a projective module $P_0^{\oplus f_{0,0}}$ where P_0 is as in § 2 and $f_{0,0} = \dim \text{Hom}(G_0, \mathcal{E}(1))$. Hence we see that

$$E_2^{p,0} = \begin{cases} \mathcal{O}^{\oplus f_{0,0}} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

Thus we infer that

$$E_\infty^{p,q} = \begin{cases} E_3^{-2,1} = \text{Ker}(\mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus f_{0,0}}) & \text{if } (p, q) = (-2, 1) \\ E_3^{0,0} = \text{Coker}(\mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus f_{0,0}}) & \text{if } (p, q) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

The Bondal spectral sequence then shows that

$$E_\infty^{p,q} = \begin{cases} \mathcal{E} & \text{if } (p, q) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus f_{0,0}} \rightarrow \mathcal{E} \rightarrow 0.$$

By looking at ranks, we see that $f_{0,0} = r + 1$, and we get the case (3) of the theorem. \square

Remark 7.2. *In the statement of [16, §2 Lemma 1] in case $(a, b) = (1, 1)$, the case (3), where $d_{\min} = 1$, in Theorem 7.1 is missing, and, instead, “the restriction of a spinor bundles from \mathbb{Q}^3 ” is added. Since the restriction of a spinor bundles from \mathbb{Q}^3 is $\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$, where $d_{\min} = 0$, this is an error. However one can understand that the case (3) in Theorem 7.1 would be what they actually wanted to say by the terms “the restriction of a spinor bundles from \mathbb{Q}^3 ” if one read through [21, §3].*

8. RESULTS ON SPINOR BUNDLES

In this section, we assume that the base field K is of characteristic zero, and recall some results on spinor bundles. Although we do not follow his convention for “spinor bundles”, Ottaviani’s results in [15] is very useful in this paper. We rephrase his results under Kapranov’s convention for later use. Throughout this section, let \mathcal{S} denote the (spanned) spinor bundle on an odd-dimensional smooth hyperquadric \mathbb{Q}^n , and \mathcal{S}^+ and \mathcal{S}^- the (spanned) spinor bundles on an even-dimensional smooth hyperquadric \mathbb{Q}^n . Besides that the sequences (G_0, \dots, G_m) in the cases (2) and (3) of § 3 are strong and exceptional, all the results we need about spinor bundles are summarized in the following theorem.

Theorem 8.1. *Set $s = \lfloor \frac{n-1}{2} \rfloor$ and let H be a smooth hyperplane section of \mathbb{Q}^n . Then we have the following.*

- (0) $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ are (spanned) spinor bundles on \mathbb{Q}^2 .
- (1) $\mathcal{S}^+|_H \cong \mathcal{S}$ and $\mathcal{S}^-|_H \cong \mathcal{S}$.
- (2) $H^0(\mathbb{Q}^n, \mathcal{S}^+) \cong H^0(H, \mathcal{S})$ and $H^0(\mathbb{Q}^n, \mathcal{S}^-) \cong H^0(H, \mathcal{S})$.
- (3) $\mathcal{S}|_H \cong \mathcal{S}^+ \oplus \mathcal{S}^-$.
- (4) $H^0(\mathbb{Q}^n, \mathcal{S}) \cong H^0(H, \mathcal{S}^+ \oplus \mathcal{S}^-)$.
- (5) $\text{rank } \mathcal{S} = 2^s$, $\dim H^0(\mathcal{S}) = 2^{s+1}$, and $\det \mathcal{S} = \mathcal{O}(2^{s-1})$.
- (6) $\text{rank } \mathcal{S}^+ = 2^s = \text{rank } \mathcal{S}^-$ and $\dim H^0(\mathcal{S}^+) = 2^{s+1} = \dim H^0(\mathcal{S}^-)$.
- (7) $\det \mathcal{S}^+ = \mathcal{O}(2^{s-1}) = \det \mathcal{S}^-$ if $s \geq 1$.
- (8) \mathcal{S} , \mathcal{S}^+ , and \mathcal{S}^- are all μ -stable bundles (with respect to any ample line bundle).

- (9) $\mathcal{S}^\vee \cong \mathcal{S}(-1)$.
- (10) $(\mathcal{S}^+)^\vee \cong \mathcal{S}^+(-1)$ and $(\mathcal{S}^-)^\vee \cong \mathcal{S}^-(-1)$ and if s is odd.
- (11) $(\mathcal{S}^+)^\vee \cong \mathcal{S}^-(-1)$ and $(\mathcal{S}^-)^\vee \cong \mathcal{S}^+(-1)$ and if s is even.

Proof. Note that our spinor bundles are the duals of those of Ottaviani's. The statement of (0) is, e.g., in [15, Example 1.5], and already used in this paper. (1) and (3) follow from [15, Theorem 1.4]. (2) and (4) follow from (1), (3), and [15, Theorem 2.3] (or Bott's vanishing theorem). (5) and (6) follow from (0), (1), (2), (3), and (4). (7) follows from (0), (1), and (3). A theorem of Ramanan [20] and Umemura [22, Theorem (2.4)] shows (8). Finally (9), (10), and (11) follow from [15, Theorem 2.8], since $n = 2(s+1)$ if n is even. \square

Lemma 8.2. *We have the following isomorphisms.*

- (1) If n is odd, then $\mathrm{RHom}(\mathcal{S}(1), \mathcal{S}) \cong K[-1]$.
- (2) If n is even, then

$$\begin{aligned} \mathrm{RHom}(\mathcal{S}^+(1), \mathcal{S}^+) &\cong 0, & \mathrm{RHom}(\mathcal{S}^-(1), \mathcal{S}^-) &\cong 0, \\ \mathrm{RHom}(\mathcal{S}^+(1), \mathcal{S}^-) &\cong K[-1], & \mathrm{RHom}(\mathcal{S}^-(1), \mathcal{S}^+) &\cong K[-1]. \end{aligned}$$

In particular, the following isomorphisms hold.

$$\mathrm{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{S}^+) \cong K[-1], \quad \mathrm{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{S}^-) \cong K[-1]$$

Proof. (1) Suppose that n is odd. Then $\mathcal{S}|_H \cong \mathcal{S}^+ \oplus \mathcal{S}^-$ for a smooth hyperplane section H of \mathbb{Q}^n by Theorem 8.1. We have the following distinguished triangle

$$\mathrm{RHom}(\mathcal{S}(1), \mathcal{S}) \rightarrow \mathrm{RHom}(\mathcal{S}, \mathcal{S}) \rightarrow \mathrm{RHom}(\mathcal{S}^+ \oplus \mathcal{S}^-, \mathcal{S}^+ \oplus \mathcal{S}^-) \rightarrow .$$

Since $\mathrm{RHom}(\mathcal{S}, \mathcal{S}) \cong K$ and $\mathrm{RHom}(\mathcal{S}^+ \oplus \mathcal{S}^-, \mathcal{S}^+ \oplus \mathcal{S}^-) \cong K \oplus K$, we get a distinguished triangle

$$\mathrm{RHom}(\mathcal{S}(1), \mathcal{S}) \rightarrow K \rightarrow K \oplus K \rightarrow .$$

Since \mathcal{S} is μ -stable by Theorem 8.1, we have $\mathrm{Hom}(\mathcal{S}(1), \mathcal{S}) = 0$. Therefore we conclude that $\mathrm{RHom}(\mathcal{S}(1), \mathcal{S}) \cong K[-1]$.

(2) Suppose that n is even. Then $\mathcal{S}^+|_H \cong \mathcal{S}$ and $\mathcal{S}^-|_H \cong \mathcal{S}$ for a smooth hyperplane section H of \mathbb{Q}^n by Theorem 8.1. We have the following distinguished triangle

$$\mathrm{RHom}(\mathcal{S}^+(1), \mathcal{S}^+) \rightarrow \mathrm{RHom}(\mathcal{S}^+, \mathcal{S}^+) \rightarrow \mathrm{RHom}(\mathcal{S}, \mathcal{S}) \rightarrow .$$

Since $K \cong \mathrm{RHom}(\mathcal{S}^+, \mathcal{S}^+) \rightarrow \mathrm{RHom}(\mathcal{S}, \mathcal{S}) \cong K$ is isomorphic, we see that

$$\mathrm{RHom}(\mathcal{S}^+(1), \mathcal{S}^+) \cong 0.$$

By the similar argument, we get $\mathrm{RHom}(\mathcal{S}^-(1), \mathcal{S}^-) \cong 0$. We have the following distinguished triangle

$$\mathrm{RHom}(\mathcal{S}^+(1), \mathcal{S}^-) \rightarrow \mathrm{RHom}(\mathcal{S}^+, \mathcal{S}^-) \rightarrow \mathrm{RHom}(\mathcal{S}, \mathcal{S}) \rightarrow .$$

Since $\mathrm{RHom}(\mathcal{S}^+, \mathcal{S}^-) \cong 0$, we see that $\mathrm{RHom}(\mathcal{S}^+(1), \mathcal{S}^-) \cong K[-1]$. By the similar argument, we get $\mathrm{RHom}(\mathcal{S}^-(1), \mathcal{S}^+) \cong K[-1]$ since $\mathrm{RHom}(\mathcal{S}^-, \mathcal{S}^+) \cong 0$. \square

9. THE CASE WHERE X IS A SMOOTH HYPERQUADRIC

Let $X, G_0, \dots, G_m, \mathcal{E}, \mathcal{O}_X(1), d_{\min}, e_{l,j}$, and \mathcal{P}_l be as in § 2 for $0 \leq j \leq l \leq m$. Assume that \mathcal{E} be a nef vector bundle of rank r as in § 3. In this section, we assume that the base field K is of characteristic zero, and that X, G_0, \dots, G_m , and $\mathcal{O}_X(1)$ are as in the cases (2) and (3) in § 3 with $n \geq 3$. In particular, X is a smooth hyperquadric \mathbb{Q}^n of dimension $n \geq 3$. Let d be as in § 3.

Lemma 9.1. *Suppose that $d = 1$ and that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$. Let \mathbb{Q}^2 be a linear section of dimension two of \mathbb{Q}^n . Then $\text{Hom}(\mathcal{O}_{\mathbb{Q}^2}, \mathcal{E}|_{\mathbb{Q}^2}) \cong \text{Hom}(\mathcal{O}, \mathcal{E})$.*

Proof. We have a distinguished triangle

$$\text{RHom}(\mathcal{O}(k), \mathcal{E}(1)) \rightarrow \text{RHom}(\mathcal{O}(k-1), \mathcal{E}(1)) \rightarrow \text{RHom}(\mathcal{O}_H(k-1), \mathcal{E}|_H(1)) \rightarrow$$

for a hyperplane section H of \mathbb{Q}^n and a integer k . Since $\text{Hom}(\mathcal{O}(2), \mathcal{E}(1)) = 0$ by assumption and $d_{\min} \leq 1$ by Corollary 4.4, we have $\text{RHom}(\mathcal{O}(k), \mathcal{E}(1)) = 0$ for $2 \leq k \leq n-1$. Therefore $\text{RHom}(\mathcal{O}(k-1), \mathcal{E}(1)) \cong \text{RHom}(\mathcal{O}_H(k-1), \mathcal{E}|_H(1))$ for $1 \leq k-1 \leq n-2$. Hence we see that $\text{Hom}(\mathcal{O}(1), \mathcal{E}(1)) \cong \text{Hom}(\mathcal{O}_H(1), \mathcal{E}|_H(1))$ and that if $n \geq 4$ then $\text{Hom}(\mathcal{O}_H(2), \mathcal{E}|_H(1))$ is zero. Now we obtain the desired formulas by induction. \square

Proposition 9.2. *Suppose that $d = 1$ and that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$. If $\text{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$ for a spinor bundle \mathcal{S} , then $n = 3$ or 4 , and $\mathcal{E} \cong \mathcal{S} \oplus \mathcal{O}^{\oplus r-2}$.*

Proof. Let $\varphi : \mathcal{S} \rightarrow \mathcal{E}$ be a non-zero element of $\text{Hom}(\mathcal{S}, \mathcal{E})$.

Suppose that $\varphi|_H = 0$ for some smooth hyperplane section H of \mathbb{Q}^n . Then we have $\text{Hom}(\mathcal{S}, \mathcal{E}(-1)) \neq 0$; let ψ be a non-zero element of $\text{Hom}(\mathcal{S}, \mathcal{E}(-1))$. We have $\psi|_L \neq 0$ for a general 2-dimensional linear section L of \mathbb{Q}^n . Note here that $\mathcal{S}|_L \cong (\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))^{\oplus s}$, where $s = \lfloor \frac{n-1}{2} \rfloor$, by Theorem 8.1. Hence $H^0((\mathcal{S}|_L)^\vee \otimes \mathcal{E}|_L(-1)) = 0$ by Theorem 7.1. This contradicts the fact that $\psi|_L \neq 0$. Hence $\varphi|_H \neq 0$ for any smooth hyperplane section H of \mathbb{Q}^n . Since the restriction of a spinor bundle to a smooth hyperplane section is again a spinor bundle or a direct sum of spinor bundles by Theorem 8.1, the argument above implies, by induction, that $\varphi|_{\mathbb{Q}^2} \neq 0$ for any 2-dimensional smooth linear section \mathbb{Q}^2 of \mathbb{Q}^n .

Denote by \mathcal{Q} the image $\text{Im}(\varphi)$ of φ and by \mathcal{F} the cokernel $\text{Coker}(\varphi)$ of φ . Let D be the singular locus of \mathcal{F} , i.e., let its complement $X \setminus D$ be the set of points at which \mathcal{F} is locally free. Let E be the singular locus of \mathcal{Q} . Then E is contained in D . Since \mathcal{Q} is torsion-free, E has codimension ≥ 2 . Note that for each point x in \mathbb{Q}^n we can take a smooth 2-dimensional linear section L of \mathbb{Q}^n such that L contains x , that L is not contained in D , and that $L \cap E$ has codimension ≥ 2 in L . We have a surjection $\mathcal{Q}|_L \rightarrow \text{Im}(\varphi|_L)$. On the other hand, $\mathcal{Q}|_{L \setminus D} \rightarrow \mathcal{E}|_{L \setminus D}$ is injective. Since $\mathcal{Q}|_{L \setminus E}$ is torsion free, we see that $\mathcal{Q}|_{L \setminus E} \rightarrow \mathcal{E}|_{L \setminus E}$ is injective. Hence $(\mathcal{Q}|_L)|_{L \setminus E} \rightarrow \text{Im}(\varphi|_L)|_{L \setminus E}$ is injective, and therefore $(\mathcal{Q}|_L)|_{L \setminus E} \rightarrow \text{Im}(\varphi|_L)|_{L \setminus E}$ is an isomorphism.

By Theorem 8.1, we see that $\det \mathcal{S} \cong \mathcal{O}(2^{s-1})$, that $\text{rank } \mathcal{S} = 2^s$, and that \mathcal{S} is μ -stable with respect to $\mathcal{O}(1)$. We have $1 = \deg \mathcal{S} / \text{rank } \mathcal{S} \leq \deg \mathcal{Q} / \text{rank } \mathcal{Q}$, since the degree $\deg \mathcal{S}$ of \mathcal{S} with respect to $\mathcal{O}(1)$ is $(\det \mathcal{S}) \cdot \mathcal{O}(1)^{n-1} = 2^s$.

The existence of $\varphi|_L \neq 0$ implies that $\mathcal{E}|_L$ is isomorphic to either $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1}$ or $\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1) \oplus \mathcal{O}^{\oplus r-2}$ by Theorem 7.1.

Suppose that $\mathcal{E}|_L$ is isomorphic $\mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1}$. Since $\mathcal{S}|_L \cong (\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))^{\oplus s}$, $\text{Im}(\varphi|_L)$ is a subsheaf of a subsheaf $\mathcal{O}_L(1)$ of $\mathcal{E}|_L$. Since $\text{Im}(\varphi|_L)|_{L \setminus E}$ is isomorphic to $\mathcal{Q}|_{L \setminus E}$, we see that $\text{rank } \mathcal{Q} = 1$. Let $\mathcal{Q}^{\vee\vee}$ be the reflexive hull of \mathcal{Q} . Then $\mathcal{Q}^{\vee\vee}$ is a line

bundle and it is a subsheaf of \mathcal{E} . Since $1 \leq \deg \mathcal{Q} = \deg \mathcal{Q}^{\vee\vee}$, this implies that \mathcal{E} contains $\mathcal{O}(1)$ as a subsheaf. This contradicts the assumption that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$. Therefore this case does not occur.

Suppose that $\mathcal{E}|_L \cong \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1) \oplus \mathcal{O}^{\oplus r-2}$. Since $\mathcal{S}|_L \cong (\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))^{\oplus s}$, $\text{Im}(\varphi|_L)$ is either one of $\mathcal{O}(1, 0)$, $\mathcal{O}(0, 1)$, or $\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$. Since $\det \text{Im}(\varphi|_L)|_{L \setminus E}$ is isomorphic to $\det \mathcal{Q}|_{L \setminus E}$, we see that a morphism $(\det \mathcal{Q})|_L \rightarrow \det \text{Im}(\varphi|_L)$ of line bundles is surjective in codimension two. Therefore $(\det \mathcal{Q})|_L \rightarrow \det \text{Im}(\varphi|_L)$ is an isomorphism. Since $\det \text{Im}(\varphi|_L)$ is thus the restriction of the line bundle $\det \mathcal{Q}$ on \mathbb{Q}^n , we conclude that $\text{Im}(\varphi|_L) \cong \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$. Thus \mathcal{Q} has rank two and $\det \mathcal{Q} \cong \mathcal{O}(1)$. Hence $\deg \mathcal{Q} / \text{rank } \mathcal{Q} = 1$. Therefore we have $\deg \mathcal{S} / \text{rank } \mathcal{S} = \deg \mathcal{Q} / \text{rank } \mathcal{Q}$, which implies that φ is an isomorphism onto its image \mathcal{Q} . Thus $s = 1$, i.e., $n = 3$ or 4 . Since \mathcal{Q} is now a vector bundle, so is $\mathcal{Q}|_L$. Since two vector bundles $\mathcal{Q}|_L$ and $\text{Im}(\varphi|_L)$ are isomorphic in codimension one, we see that $\mathcal{Q}|_L$ and $\text{Im}(\varphi|_L)$ are isomorphic. Since $\text{Im}(\varphi|_L)$ is a subbundle of $\mathcal{E}|_L$, we conclude that \mathcal{S} is a subbundle of \mathcal{E} . Thus \mathcal{F} is a nef vector bundle with $\det \mathcal{F} \cong 0$. Hence $\mathcal{F} \cong \mathcal{O}^{\oplus r-2}$ and we obtain $\mathcal{E} \cong \mathcal{S} \oplus \mathcal{O}^{\oplus r-2}$. \square

Based on our framework, we give a different proof of the following theorem of Peternell-Szurek-Wisniewski [16, Theorem 2].

Theorem 9.3. *Suppose that $d = 1$, i.e., that $\det \mathcal{E} \cong \mathcal{O}(1)$. Then \mathcal{E} satisfies one of the following:*

- (1) $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1}$.
- (2) $\mathcal{E} \cong \mathcal{S} \oplus \mathcal{O}^{\oplus r-2}$, where \mathcal{S} is a spinor bundle and $n = 3$ or 4 .
- (3) \mathcal{E} fits in an exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow 0.$$

Proof. If $d_{\min} < 0$, then $r = 1$ by Proposition 3.1 (2) (d). If $r = 1$, then $\mathcal{E} \cong \mathcal{O}(1)$ and $d_{\min} = -1$. In the following, we assume that $d_{\min} \geq 0$ and that $r \geq 2$. We know that $d_{\min} \leq 1$ by Corollary 4.4.

If $\text{Hom}(\mathcal{O}(1), \mathcal{E}) \neq 0$, then $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus r-1}$ by Proposition 5.2. In the following, we assume that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$.

If $\text{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$, then $n = 3$ or 4 , and $\mathcal{E} \cong \mathcal{S} \oplus \mathcal{O}^{\oplus r-2}$ by Proposition 9.2. In the following, we assume that $\text{Hom}(\mathcal{S}, \mathcal{E}) = 0$.

Under the assumptions that $d_{\min} \geq 0$, that $\text{Hom}(\mathcal{O}(1), \mathcal{E}) = 0$, and that $\text{Hom}(\mathcal{S}, \mathcal{E}) = 0$, we have $d_{\min} = 1$. Indeed, if d_{\min} were zero, then \mathcal{E} would be isomorphic to $\mathcal{O}^{\oplus r}$ by the standard resolution, which contradicts that $d = 1$.

In the following, we assume $d_{\min} = 1$. Since $\text{Hom}(\mathcal{O}(2), \mathcal{E}(1)) = 0$, we see that $e_{0,3} = 0$ if n is odd, and that $e_{0,4} = 0$ if n is even. Set $e = \dim H^0(\mathcal{E})$. Then $e_{0,2} = e$ if n is odd, and $e_{0,3} = e$ if n is even. By Lemma 9.1, we have $e = \dim H^0(\mathcal{E}|_{\mathbb{Q}^2})$ for any 2-dimensional smooth linear section \mathbb{Q}^2 of \mathbb{Q}^n . Moreover we have $\dim H^0(\mathcal{E}|_{\mathbb{Q}^2}) \geq r + 1$ by Theorem 7.1. Therefore we see that

$$e \geq r + 1.$$

Note that \mathcal{E} is globally generated since $\mathcal{E}|_{\mathbb{Q}^2}$ is globally generated by Theorem 7.1 and $H^0(\mathcal{E}) \cong H^0(\mathcal{E}|_{\mathbb{Q}^2})$ by Lemma 9.1 for any \mathbb{Q}^2 . Hence we obtain the desired exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus r+1} \rightarrow \mathcal{E} \rightarrow 0$$

if $e = r + 1$.

In the following, we shall show that $e = r + 1$. Set $s = \lfloor \frac{n-1}{2} \rfloor$. We divide the case according to whether n is odd or not.

Suppose that n is odd. Then $n = 2s + 1$. The standard resolution of $\mathcal{E}(1)$ modified according to Proposition 2.7 is

$$0 \rightarrow \mathcal{O}^{\oplus e_{2,0}} \rightarrow \mathcal{S}^{\oplus e_{1,1}} \oplus \mathcal{O}^{\oplus (e_{1,0} - e_{0,0})} \rightarrow \mathcal{O}(1)^{\oplus e} \oplus \mathcal{S}^{\oplus e_{0,1}} \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Since $\text{Hom}(\mathcal{S}, \mathcal{O}(1)) \cong H^0(\mathcal{S})$ and $\dim H^0(\mathcal{S}) = 2^{s+1}$ by Theorem 8.1, we see $e_{1,1} = 2^{s+1}e$ and $e_{2,0} = 2^{2s+2}e = 2^{n+1}e$. Since $\det \mathcal{S} \cong \mathcal{O}(2^{s-1})$ by Theorem 8.1, by looking at $\det(\mathcal{E}(1))$, we see that

$$1 + r = e + 2^{s-1}(e_{0,1} - e_{1,1}) = 2^{s-1}e_{0,1} + (1 - 2^{n-1})e.$$

Hence

$$e_{0,1} = 2^{1-s}\{1 + r + (2^{n-1} - 1)e\}.$$

Since $\text{rank } \mathcal{S} = 2^s$ by Theorem 8.1, by looking at $\text{rank } \mathcal{E}(1)$, we see that

$$\begin{aligned} r &= e + 2\{1 + r + (2^{n-1} - 1)e\} - 2^n e - e_{1,0} + e_{0,0} + 2^{n+1}e \\ &= 2r + 2 + (2^{n+1} - 1)e - e_{1,0} + e_{0,0}. \end{aligned}$$

Hence $e_{1,0} - e_{0,0} = r + 2 + (2^{n+1} - 1)e$. Summing up, we have a locally free resolution

$$0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{P}_1^0 \rightarrow \mathcal{P}_0^0 \rightarrow \mathcal{E}(1) \rightarrow 0,$$

where

$$\begin{aligned} \mathcal{P}_2 &= \mathcal{O}^{\oplus 2^{n+1}e}, \\ \mathcal{P}_1^0 &= \mathcal{O}^{\oplus r+2+(2^{n+1}-1)e} \oplus \mathcal{S}^{\oplus 2^{s+1}e}, \\ \mathcal{P}_0^0 &= \mathcal{S}^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}} \oplus \mathcal{O}(1)^{\oplus e}. \end{aligned}$$

We split the above long exact sequence into the following two short exact sequences

$$0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{P}_1^0 \rightarrow \mathcal{G} \rightarrow 0, \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{P}_0^0 \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Note that $\text{RHom}(\mathcal{S}(1), \mathcal{O}(1)) \cong 0$. It follows from Theorem 8.1 that

$$\text{RHom}(\mathcal{S}(1), \mathcal{O}) \cong \text{RHom}(\mathcal{O}(1), \mathcal{S}^\vee) \cong \text{RHom}(\mathcal{O}(2), \mathcal{S}^\vee(1)) \cong \text{RHom}(\mathcal{O}(2), \mathcal{S}) \cong 0.$$

Hence $\text{RHom}(\mathcal{S}(1), \mathcal{P}_2) \cong 0$, and thus

$$\text{RHom}(\mathcal{S}(1), \mathcal{P}_1^0) \cong \text{RHom}(\mathcal{S}(1), \mathcal{G}).$$

Since $\text{RHom}(\mathcal{S}(1), \mathcal{S}) \cong K[-1]$ by Lemma 8.2, we see that

$$\begin{aligned} \text{RHom}(\mathcal{S}(1), \mathcal{P}_0^0) &\cong K[-1]^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}}, \\ \text{RHom}(\mathcal{S}(1), \mathcal{P}_1^0) &\cong K[-1]^{\oplus 2^{s+1}e}. \end{aligned}$$

Therefore we obtain a distinguished triangle

$$K[-1]^{\oplus 2^{s+1}e} \rightarrow K[-1]^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}} \rightarrow \text{RHom}(\mathcal{S}(1), \mathcal{E}(1)) \rightarrow .$$

Since we assume now that $\text{Hom}(\mathcal{S}, \mathcal{E}) = 0$, we get an exact sequence

$$0 \rightarrow K^{\oplus 2^{s+1}e} \rightarrow K^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}} \rightarrow \text{Ext}^1(\mathcal{S}(1), \mathcal{E}(1)) \rightarrow 0.$$

In particular we have $2^{s+1}e \leq 2^{1-s}\{1 + r + (2^{n-1} - 1)e\}$, i.e., $e \leq r + 1$. Hence

$$e = r + 1.$$

Suppose that n is even. Then $n = 2s + 2$. The standard resolution of $\mathcal{E}(1)$ modified according to Proposition 2.7 is

$$0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{P}_1^0 \rightarrow \mathcal{P}_0^0 \rightarrow \mathcal{E}(1) \rightarrow 0,$$

where

$$\begin{aligned}\mathcal{P}_2 &= \mathcal{O}^{\oplus e_{2,0}}, \\ \mathcal{P}_1^0 &= \mathcal{O}^{\oplus e_{1,0} - e_{0,0}} \oplus (\mathcal{S}^+)^{\oplus e_{1,1}} \oplus (\mathcal{S}^-)^{\oplus e_{1,2}}, \\ \mathcal{P}_0^0 &= (\mathcal{S}^+)^{\oplus e_{0,1}} \oplus (\mathcal{S}^-)^{\oplus e_{0,2}} \oplus \mathcal{O}(1)^{\oplus e}.\end{aligned}$$

By Theorem 8.1, we see that $\text{Hom}(\mathcal{S}^+, \mathcal{O}(1)) \cong H^0(\mathcal{S}^-)$ and $\text{Hom}(\mathcal{S}^-, \mathcal{O}(1)) \cong H^0(\mathcal{S}^+)$ if s is even, and that $\text{Hom}(\mathcal{S}^+, \mathcal{O}(1)) \cong H^0(\mathcal{S}^+)$ and $\text{Hom}(\mathcal{S}^-, \mathcal{O}(1)) \cong H^0(\mathcal{S}^-)$ if s is odd. In the following, we denote \mathcal{S}^+ and \mathcal{S}^- simply by \mathcal{S} if no confusion occurs. Since $\dim H^0(\mathcal{S}) = 2^{s+1}$ by Theorem 8.1, we see that $e_{1,1} = 2^{s+1}e$, that $e_{1,2} = 2^{s+1}e$, and that $e_{2,0} = 2^{2s+3}e = 2^{n+1}e$. Since $\det \mathcal{S} \cong \mathcal{O}(2^{s-1})$ by Theorem 8.1, by looking at $\det(\mathcal{E}(1))$, we see that

$$1 + r = e + 2^{s-1}(e_{0,1} + e_{0,2} - e_{1,1} - e_{1,2}) = 2^{s-1}(e_{0,1} + e_{0,2}) + (1 - 2^{n-1})e.$$

Hence

$$e_{0,1} + e_{0,2} = 2^{1-s}\{1 + r + (2^{n-1} - 1)e\}.$$

Since $\text{rank } \mathcal{S} = 2^s$ by Theorem 8.1, by looking at $\text{rank } \mathcal{E}(1)$, we see that

$$\begin{aligned}r &= e + 2\{1 + r + (2^{n-1} - 1)e\} - 2^n e - e_{1,0} + e_{0,0} + 2^{n+1}e \\ &= 2r + 2 + (2^{n+1} - 1)e - e_{1,0} + e_{0,0}.\end{aligned}$$

Hence $e_{1,0} - e_{0,0} = r + 2 + (2^{n+1} - 1)e$. Summing up, we have a locally free resolution

$$0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{P}_1^0 \rightarrow \mathcal{P}_0^0 \rightarrow \mathcal{E}(1) \rightarrow 0,$$

where

$$\begin{aligned}\mathcal{P}_2 &= \mathcal{O}^{\oplus 2^{n+1}e}, \\ \mathcal{P}_1^0 &= \mathcal{O}^{\oplus r+2+(2^{n+1}-1)e} \oplus (\mathcal{S}^+)^{\oplus 2^{s+1}e} \oplus (\mathcal{S}^-)^{\oplus 2^{s+1}e}, \\ \mathcal{P}_0^0 &= (\mathcal{S}^+)^{\oplus e_{0,1}} \oplus (\mathcal{S}^-)^{\oplus e_{0,2}} \oplus \mathcal{O}(1)^{\oplus e}, \quad e_{0,1} + e_{0,2} = 2^{1-s}\{1 + r + (2^{n-1} - 1)e\}.\end{aligned}$$

We split the above long exact sequence into the following two short exact sequences

$$0 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{P}_1^0 \rightarrow \mathcal{G} \rightarrow 0, \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{P}_0^0 \rightarrow \mathcal{E}(1) \rightarrow 0.$$

Note that $\text{RHom}(\mathcal{S}(1), \mathcal{O}(1)) \cong 0$ and that $\text{RHom}(\mathcal{S}(1), \mathcal{O}) \cong 0$ as in the odd-dimensional case. Hence $\text{RHom}(\mathcal{S}(1), \mathcal{P}_2) \cong 0$, and thus $\text{RHom}(\mathcal{S}(1), \mathcal{P}_1^0) \cong \text{RHom}(\mathcal{S}(1), \mathcal{G})$. Since both $\text{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{S}^+)$ and $\text{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{S}^-)$ are isomorphic to $K[-1]$ by Lemma 8.2, we see that

$$\begin{aligned}\text{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{P}_0^0) &\cong K[-1]^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}}, \\ \text{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{P}_1^0) &\cong K[-1]^{\oplus 2^{s+2}e}.\end{aligned}$$

Therefore we obtain a distinguished triangle

$$K[-1]^{\oplus 2^{s+2}e} \rightarrow K[-1]^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}} \rightarrow \text{RHom}((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{E}(1)) \rightarrow .$$

Since we assume now that $\text{Hom}(\mathcal{S}^+ \oplus \mathcal{S}^-, \mathcal{E}) = 0$, we get an exact sequence

$$0 \rightarrow K^{\oplus 2^{s+2}e} \rightarrow K^{\oplus 2^{1-s}\{1+r+(2^{n-1}-1)e\}} \rightarrow \text{Ext}^1((\mathcal{S}^+ \oplus \mathcal{S}^-)(1), \mathcal{E}(1)) \rightarrow 0.$$

In particular we have $2^{s+2}e \leq 2^{1-s}\{1+r+(2^{n-1}-1)e\}$, i.e., $e \leq r+1$. Hence

$$e = r + 1.$$

□

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GRADUATE SCHOOL OF INFORMATICS AND ENGINEERING, THE UNIVERSITY OF ELECTRO-COMMUNICATIONS,
CHOFU-SHI, TOKYO, 182-8585 JAPAN
E-mail address: `masahiro-ohno@uec.ac.jp`